Chapter 5

The two-body problem

CHAPTER CONTENT

THE TWO-BODY PROBLEM

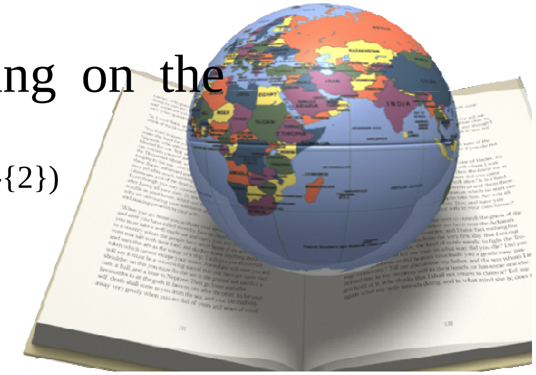
4-THE TWO-BODY PROBLEM

✶**N**ow that we have a general expression for the relative motion of two bodies perturbed by other bodies it would be a simple matter to reduce it to an equation for only two bodies.

✶ There are two assumptions we will make with regard to our model:

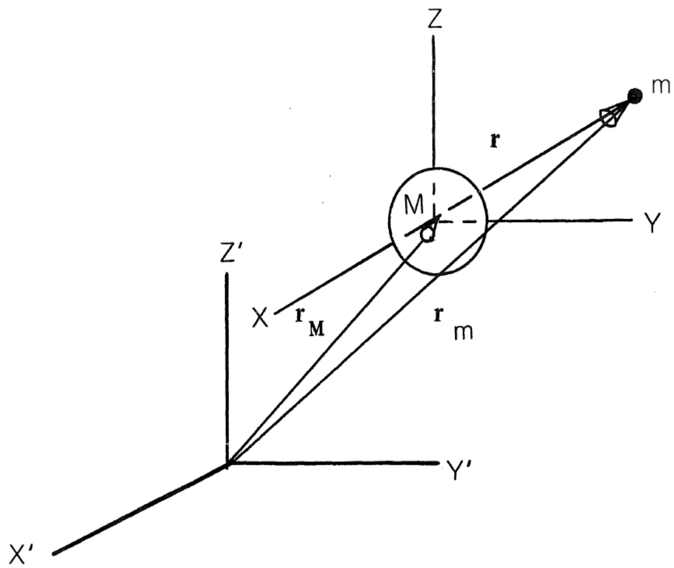
1- The bodies are spherically symmetric (Note 3-page11-{2})

2- There are no external non internal forces acting on the system other than the gravitational forces (Note 4-page12-{2})

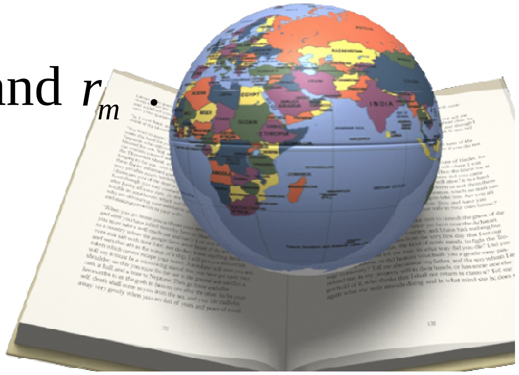


4-THE TWO-BODY PROBLEM

- (Note 5 Page12 {2})
- Consider the system of two bodies of mass M and m
- Let (x', y', z') be an internal set of rectangular cartesian coordinates.



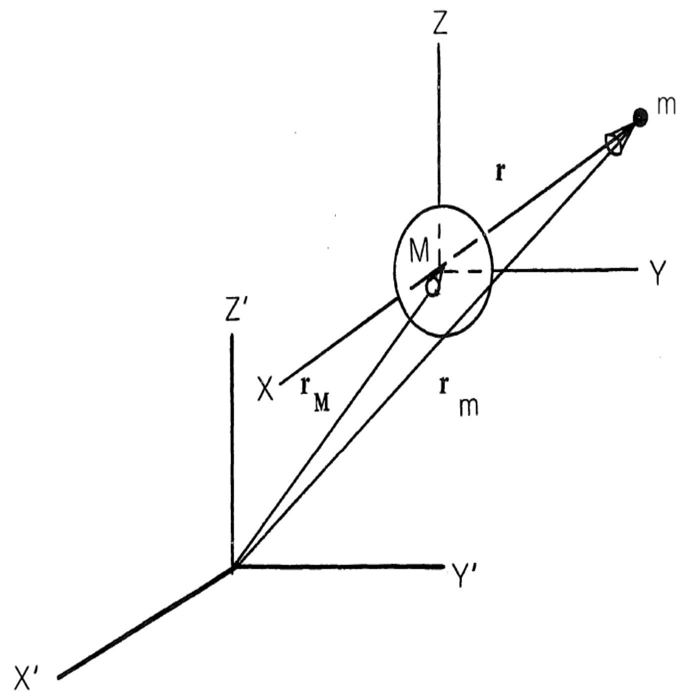
- Let (x', y', z') be a set of nonrotating coordinates parallel to (x, y, z) and having an origin coincident with the body of mass M.
- The position vectors of the bodies M and m are r_M and r_m.



4-THE TWO-BODY PROBLEM

- ✶ We have defined

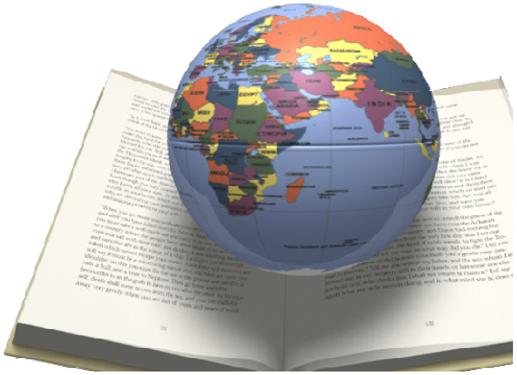
$$\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M .$$



- ✶ By applying Newton's laws in the inertial frame (x', y', z') we will obtain:

$$m\ddot{\mathbf{r}}_m = -\frac{GMm}{r^2}\frac{\mathbf{r}}{r}$$

$$M\ddot{\mathbf{r}}_M = \frac{GMm}{r^2}\frac{\mathbf{r}}{r}$$



4-THE TWO-BODY PROBLEM

- The above equations may be written:

$$\ddot{\mathbf{r}}_m = -\frac{GM}{r^3}\,\mathbf{r} \qquad (1)$$

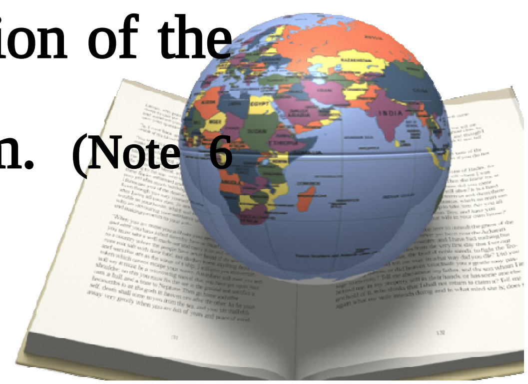
$$\ddot{\mathbf{r}}_M = \frac{Gm}{r^3}\,\mathbf{r}\,. \qquad (2)$$

- **Subtracting equation (2) from (1) we have**

$$\ddot{r} = \ddot{r}_m - \ddot{r}_M$$

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\,\mathbf{r}\,. \qquad (3)$$

- **Equation (3) is the vector differential equation of the relative motion for the two-body problem. (Note 6 Page13 {2})**



4-THE TWO-BODY PROBLEM

✶ Since our efforts will be devoted to studying the motion of satellites . Ballistic missiles or space probes orbiting about some planet or the sun, Hence we see that:

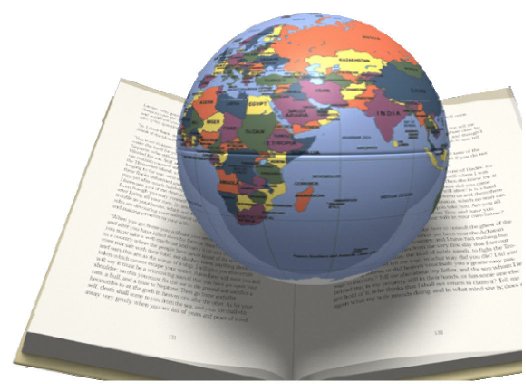
$$G(M+m) \approx GM.$$

✶ It is convenient to define a parameter μ ,called the gravitational parameter as:

$$\mu \equiv GM.$$

✶ Then the equation 3 becomes:

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = 0. \qquad (4)$$

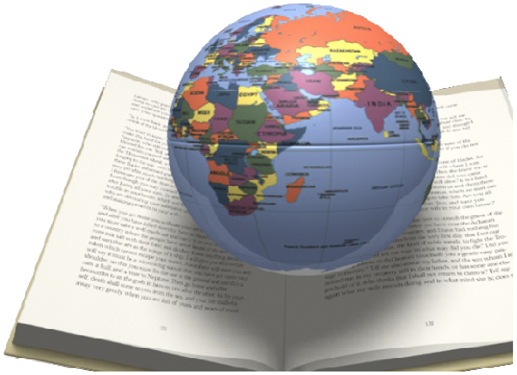


4-THE TWO-BODY PROBLEM

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = 0. \qquad (4)$$

- Equation(4) is the two-body equation of motion
- Remember the results obtained from equation(4) will be only as accurate as the assumptions (1),(2) and the assumption that $M \gg m$
- If m is not much less than M.

  then $G(M + m)$ must be used in place of μ



4-THE TWO-BODY PROBLEM

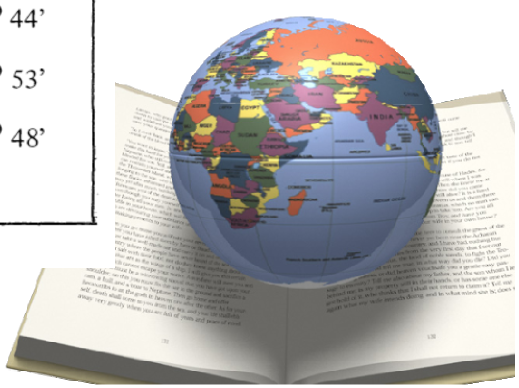
- ✶ μ will have a different value for each major attracting body

Table 8.2-3

PHYSICAL CHARACTERISTICS OF THE SUN AND PLANETS*

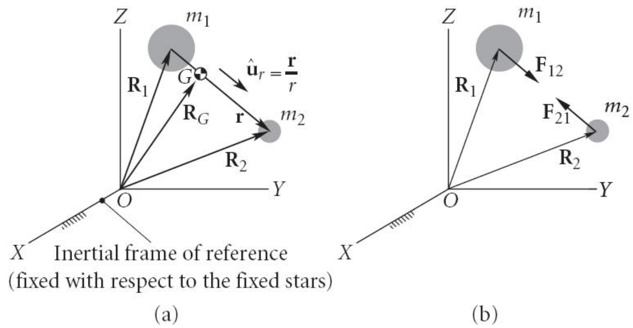
Planet	Orbital Period years	Mean distance 10^6 km	Orbital speed km/sec	Mass Earth = 1	μ km^3/sec^2	Equatorial radius km	Inclination of equator to orbit
Sun	–	–	–	333432	1.327×10^{11}	696000	7^{o} 15'
Mercury	.241	57.9	47.87	.056	2.232×10^{4}	2487	?
Venus	.615	108.1	35.04	.817	3.257×10^{5}	6187	32^{o}
Earth	1.000	149.5	29.79	1.000	3.986×10^{5}	6378	23^{o} 27'
Mars	1.881	227.8	24.14	.108	4.305×10^{4}	3380	23^{o} 59'
Jupiter	11.86	778	13.06	318.0	1.268×10^{8}	71370	3^{o} 04'
Saturn	29.46	1426	9.65	95.2	3.795×10^{7}	60400	26^{o} 44'
Uranus	84.01	2868	6.80	14.6	5.820×10^{6}	23530	97^{o} 53'
Neptune	164.8	4494	5.49	17.3	6.896×10^{6}	22320	28^{o} 48'
Pluto	247.7	5896	4.74	.9?	3.587×10^{5}?	7016?	?

*From reference 3



4-THE TWO-BODY PROBLEM

Equations of motion in an inertial frame:

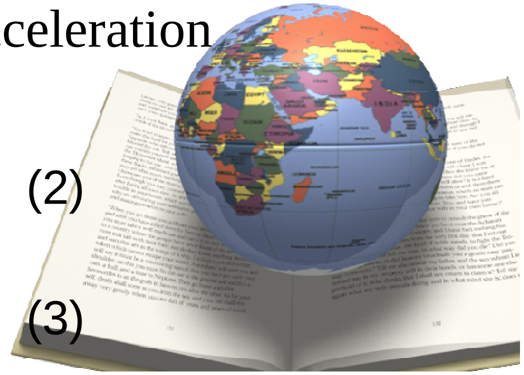


- Above figure shows two point masses acted upon only by the force of gravity between them. (Note 7. P 34.{1})

- The position vector R_G of the center of mass G of the system is defined by the formula:

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2} \qquad (1)$$

- Therefore the absolute velocity and the absolute acceleration of G are: (Note 8. P 35.{1})



$$\mathbf{v}_G = \dot{\mathbf{R}}_G = \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2} \qquad (2)$$

$$\mathbf{a}_G = \ddot{\mathbf{R}}_G = \frac{m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2}{m_1 + m_2} \qquad (3)$$

4-THE TWO-BODY PROBLEM

- Let **r** be the position vector m_2 relative to m_1, then:

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \tag{4}$$

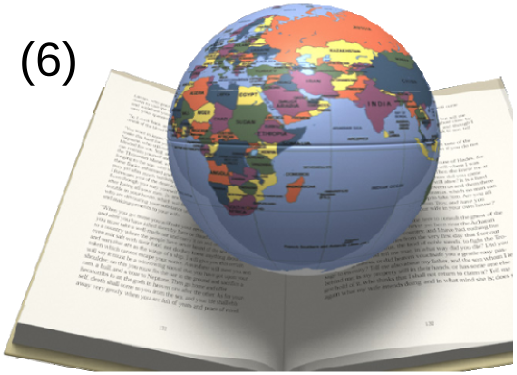
- Furthermore, let $\hat{u}_r$ be the unit vector pointing from m_1 towards m_2, so that

$$\hat{\mathbf{u}}_r = \frac{\mathbf{r}}{r} \tag{5}$$

- Where $r = \|r\|$ the magnitude of **r**

- The gravitational attraction force exerted on m_2 by m_1 is

$$\mathbf{F}_{21} = \frac{Gm_1 m_2}{r^2}(-\hat{\mathbf{u}}_r) = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{u}}_r \tag{6}$$



Note 9. P 36.{1}

4-THE TWO-BODY PROBLEM

- Newton's second law of motion as applied to body m_2 is

  $\mathbf{F}_{21} = m_2\ddot{\mathbf{R}}_2$, where $\ddot{\mathbf{R}}_2$ is the absolute acceleration of m_2

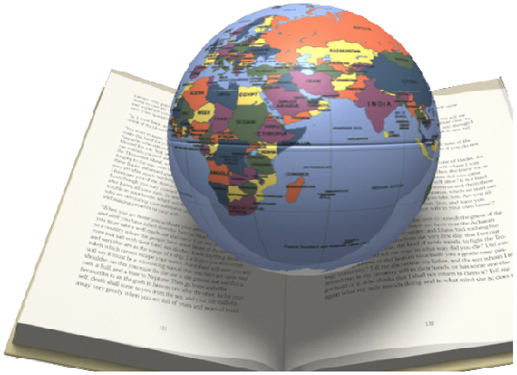
  thus:

$$-\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r = m_2\ddot{\mathbf{R}}_2 \tag{7}$$

- By Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$, so that for m_1 we have

$$\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r = m_1\ddot{\mathbf{R}}_1 \tag{8}$$

- Equations (7) and (8) are the equations of motion of the two bodies in inertial space



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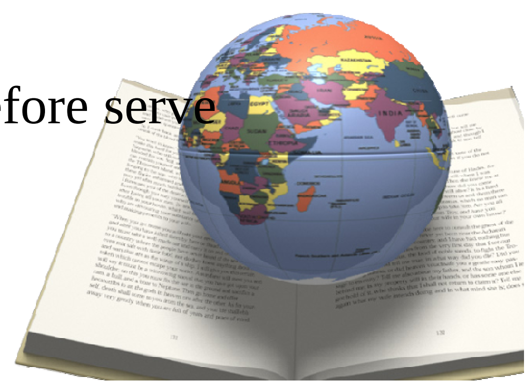
- By adding each side of these equations together we find:

$$m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2 = 0$$

- According to Equ.(3), that means the acceleration of the center of mass G of the system of two bodies m_1 and m_2 is zero.
- G moves with a constant velocity V_G in a straight lines, so that its position vector relative to XYZ given by

$$\mathbf{R}_G = \mathbf{R}_{G_0} + \mathbf{v}_G t \tag{9}$$

- Where R_{G_0} is the position of G at time $t = 0$
- The center of mass of a two-body system may therefore serve as the origin of an inertial frame.



4-THE TWO-BODY PROBLEM

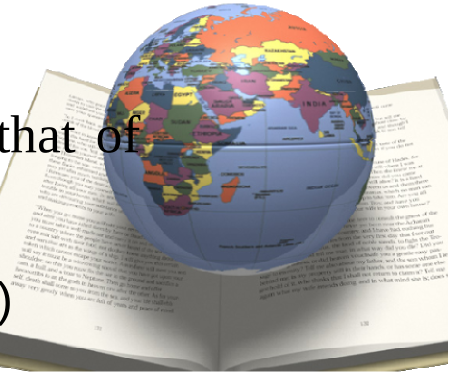
Example

Use the equations of motion to show why orbiting astronauts experience weightlessness.

Solution:

- ✶ We sense weight by feeling the contact forces that developed wherever our body is supported.
- ✶ Consider an astronaut of mass m_A strapped into the space shuttle of mass m_S in orbit about the earth.
- ✶ The distance between the center of the earth and spacecraft is **r**, and the mass of the earth is m_E
- ✶ Sense the only external force on the space shuttle is that of gravity $\mathbf{F}_S)_g$ the equation of motion of the shuttle is:

$$\mathbf{F}_S)_g = m_S\mathbf{a}_S \quad \text{(a)}$$



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EXAMPLE

- According to equation (6)

$$\mathbf{F}_S)_g = -\frac{GM_E m_S}{r^2}\hat{\mathbf{u}}_r \qquad \text{(b)}$$

$\hat{u}_r$: is the unit vector pointing outward from the earth to space shuttle.

- Thus (a) and (b) imply:

$$\mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \qquad \text{(c)}$$

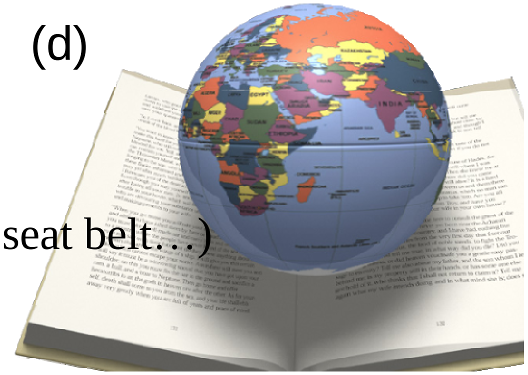
- The equation of motion of the astronaut is:

$$\mathbf{F}_A)_g + \mathbf{C}_A = m_A\mathbf{a}_A \qquad \text{(d)}$$

$F_{A)g}$:the weight of the astronaut

C_A : the net contact force on the astronaut from restraints (seat, seat belt…)

a_A : the astronaut's acceleration.



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EXAMPLE

✶ According to Equ.(6)

$$\mathbf{F}_A)_g = -\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r \qquad \text{(e)}$$

✶ Since the astronaut is moving with the shuttle we have:

$$\mathbf{a}_A = \mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \qquad \text{(f)}$$

✶ Substituting (e) and (f) into (d) yields:

$$-\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r + \mathbf{C}_A = m_A\left(-\frac{GM_E}{r^2}\hat{\mathbf{u}}_r\right)$$

✶ From which it is clear that $C_A = 0$