CHAPTER 12

PARABOLIC TRAJECTORIES

CHAPTER CONTENT

12- PARABOLIC TRAJECTORIES

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⋆**F**or the parabola (e=1) Equation:

$$\frac{\mu^2}{h^3}t = \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2}$$

⋆becomes:

$$\frac{\mu^2}{h^3}t = \int_0^{\theta} \frac{d\vartheta}{(1 + \cos\vartheta)^2} \qquad (1)$$

⋆In integral tables we find that:

$$\int_0^{\theta} \frac{d\vartheta}{(1 + \cos\vartheta)^2} = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

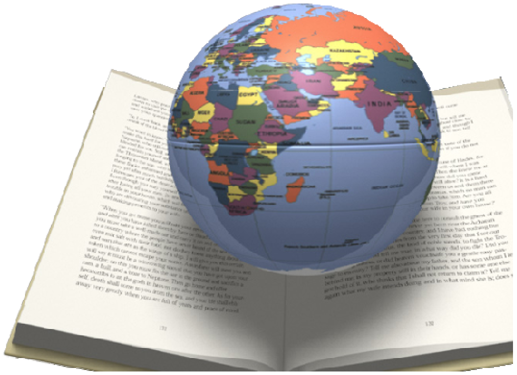
⋆Therefore equation (1) may be written as:

(Barker's Equation)

$$M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} \qquad (2)$$

⋆Where

$$M_p = \frac{\mu^2 t}{h^3} \qquad (3)$$



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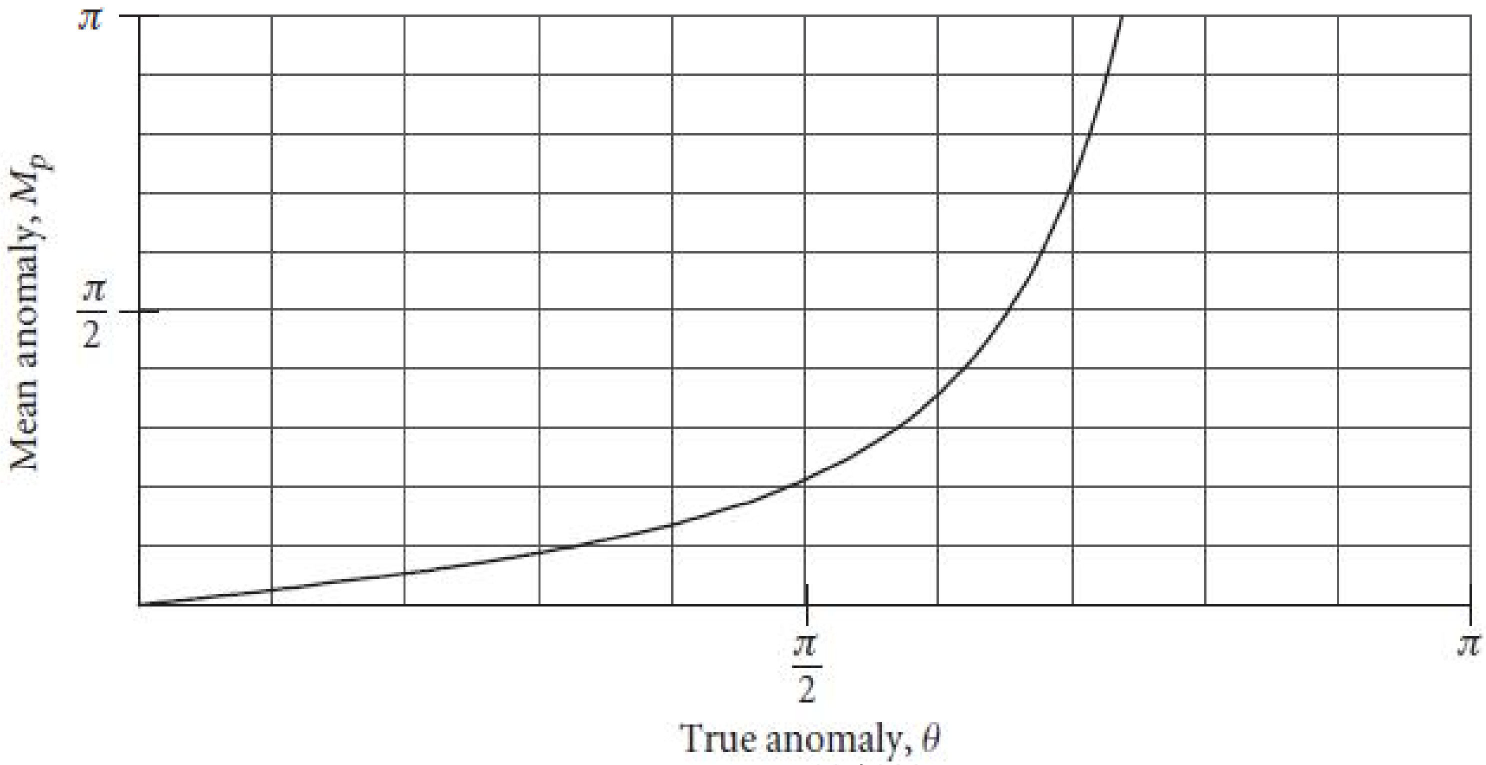
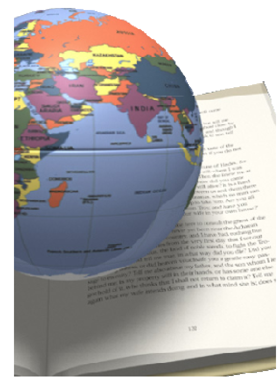
$$M_p = \frac{\mu^2 t}{h^3} \qquad (3)$$

- ✶ M_p Is dimensionless, and it may be thought of as the "mean anomaly" for the parabola.





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- Given the anomaly θ, we find the time directly from Equations (3) , (2).

$$M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

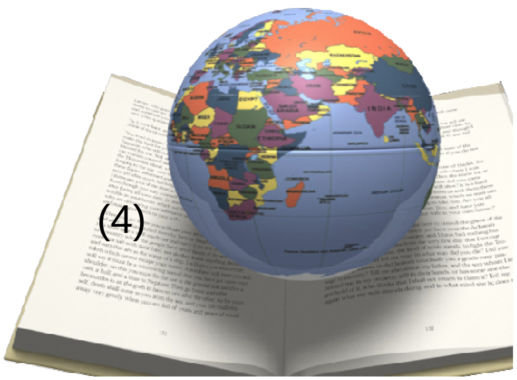
$$M_p = \frac{\mu^2 t}{h^3}$$

- If time is the given variable, then we must solve the cubic equation:

$$\frac{1}{6}\left(\tan\frac{\theta}{2}\right)^3 + \frac{1}{2}\tan\frac{\theta}{2} - M_p = 0$$

- Which has but one real root, namely:

$$\tan\frac{\theta}{2} = \left[3M_p + \sqrt{(3M_p)^2+1}\right]^{\frac{1}{3}} - \left[(3M_p + \sqrt{(3M_p)^2+1})\right]^{-\frac{1}{3}} \tag{4}$$



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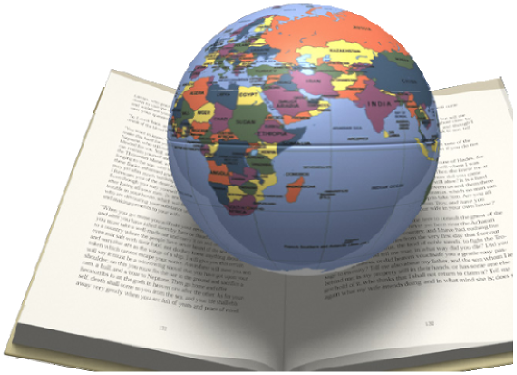
EXAMPLE 12.1

✶ A geocentric parabola has a perigee velocity of $10\ km/s$. How far is the satellite from the center of the earth six hours after perigee passage?



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EXAMPLE 12.1

- **Solution:**
- We will find the perigee radius from equation:

$$r_p = \frac{2\mu}{v_p^2} = \frac{2 \cdot 398\,600}{10^2} = 7972\,\text{km}$$

- So that the angular momentum is

$$h = r_p v_p = 7972 \cdot 10 = 79\,720\,\text{km}^2/\text{s}$$

- Now we can calculate the parabolic mean anomaly:

$$M_p = \frac{\mu^2 t}{h^3} = \frac{398\,600^2 \cdot (6 \cdot 3600)}{79\,720^3} = 6.7737\,\text{rad}$$

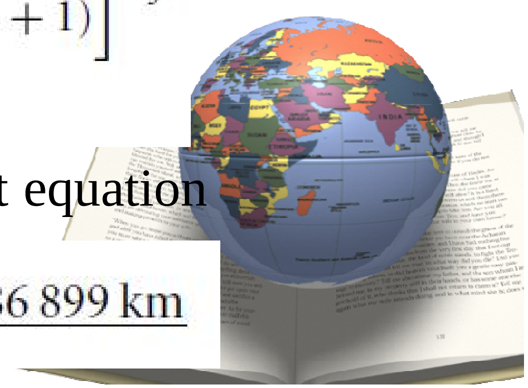
- So that $3M_p = 20.321$ rad. Equation(4) yields the true anomaly:

$$\tan\frac{\theta}{2} = \left[20.321 + \sqrt{20.321^2 + 1}\right]^{\frac{1}{3}} - \left[(20.321 + \sqrt{20.321^2 + 1})\right]^{-\frac{1}{3}}$$

$$= 3.1481 \quad \Rightarrow \quad \theta = 144.75^\circ$$

- Finally, we substitute the true anomaly into the orbit equation to find the radius:

$$r = \frac{79\,720^2}{398\,600}\frac{1}{1 + \cos(144.75^\circ)} = \underline{86\,899\,\text{km}}$$



12- HYPERBOLIC TRAJECTORIES

✶For the hyperbola (e>1) the Equation:

$$\frac{\mu^2}{h^3}(t - t_p) = \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2}$$

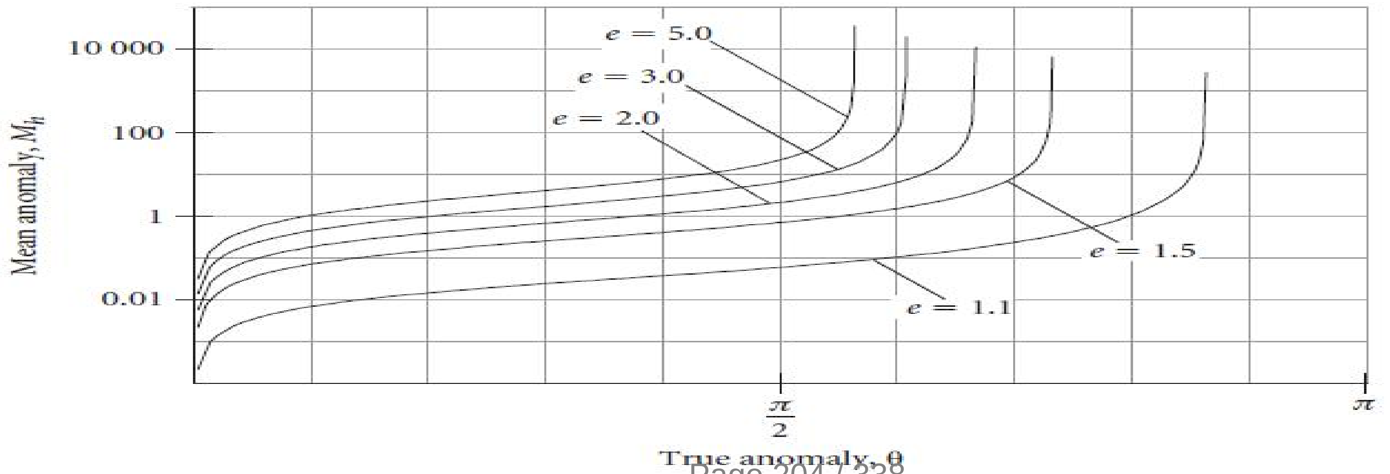
✶After some substitutions becomes:

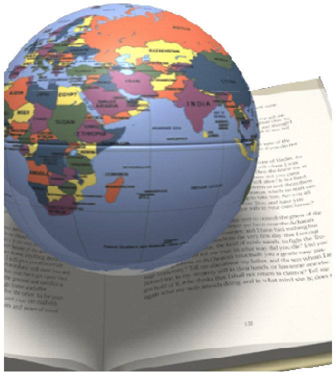
$$M_h = \frac{e\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right) \quad (1)$$

✶Where, M_h is the hyperbolic mean anomaly:

$$M_h = \frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t \quad (2)$$

✶Equation (1) is plotted in the below figure





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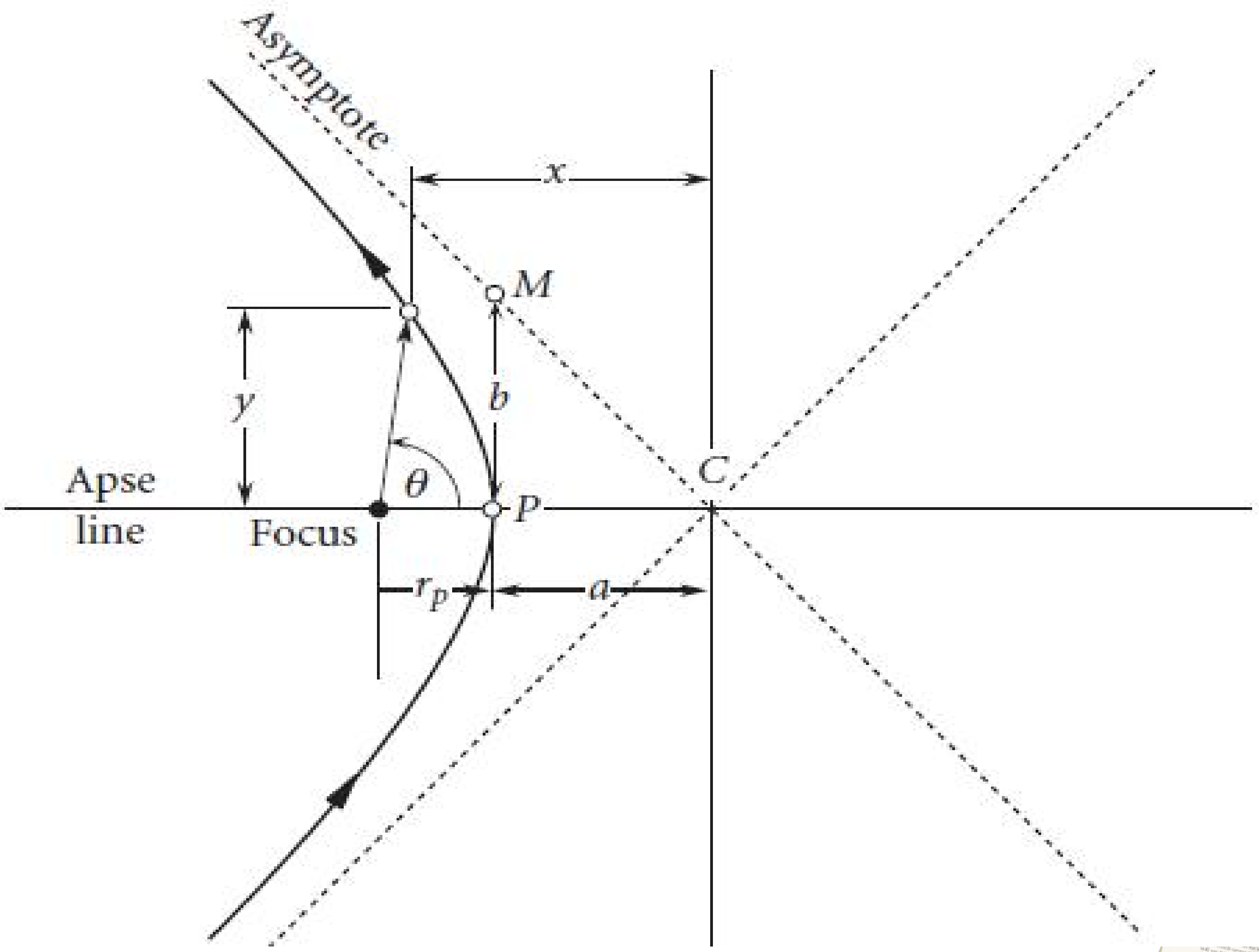
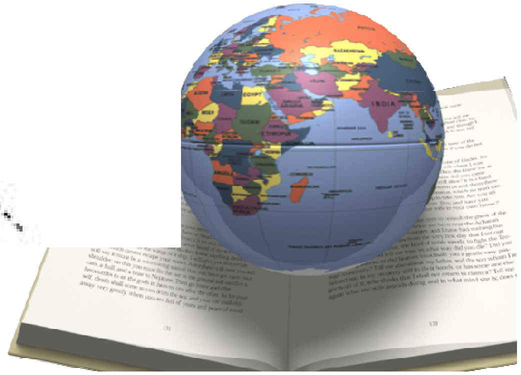


✶(NOTE 22,P126,{1})



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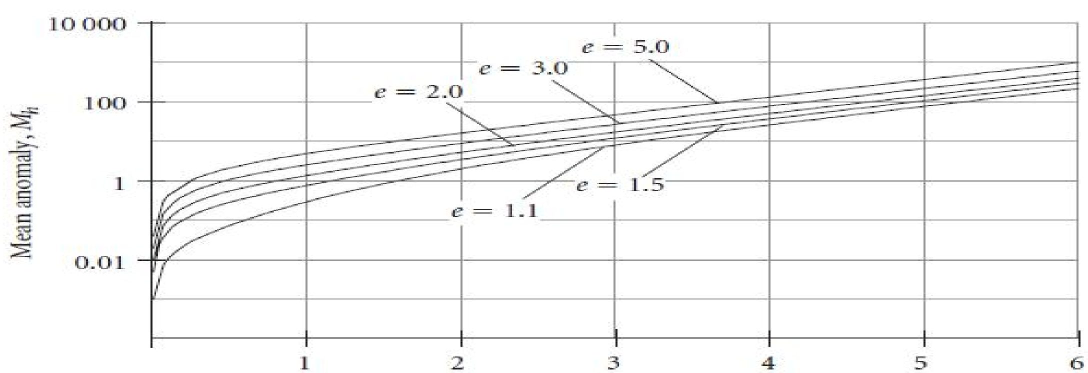
- We define F to be such that: $\sinh F = \frac{y}{b}$ (3)
- It is consistent with the definition of $\sinh F$ to define the hyperbolic cosine as: $\cosh F = \frac{x}{a}$ (4)
- We can prove that:

$$\sinh F = \frac{\sqrt{e^2 - 1}\sin\theta}{1 + e\cos\theta} \quad (5)$$

$$F = \sinh^{-1}\left(\frac{\sqrt{e^2 - 1}\sin\theta}{1 + e\cos\theta}\right) \quad (6)$$

$$F = \ln\left[\frac{\sqrt{e+1} + \sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1} - \sqrt{e-1}\tan(\theta/2)}\right] \quad (7)$$

- Substituting equation(7),(5) into equation(1), yields Kepler's equation for the hyperbola, $M_h = e\sinh F - F$ (8)
- this equation is plotted for several different eccentricities in below figure:



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- ★ If time is the given quantity, then equation(8), must be solved for F by an iterative procedure, as was the case for the ellipse $M_h = e \sinh F - F$
- ★ To apply Newton's procedure to the solution of Kepler's equation for the hyperbola, we from the function:

$$f(F) = e \sinh F - F - M_h$$

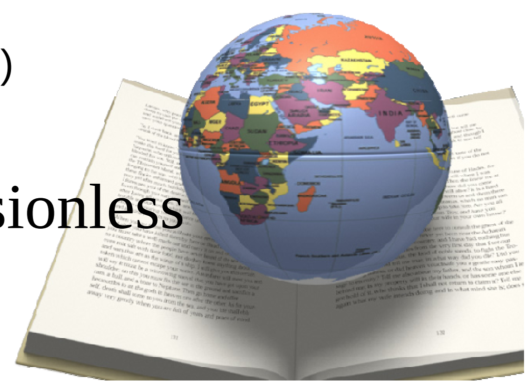
- ★ And seek the value of F that makes *f*(F)=0 since

$$f'(F) = e \cosh F - 1$$

- ★ Equation becomes

$$F_{i+1} = F_i - \frac{e \sinh F_i - F_i - M_h}{e \cosh F_i - 1} \qquad (9)$$

- ★ All quantities in this formula are dimensionless (radians, not degrees).



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- When determining orbital position as a function of time with the aid of Kepler's equation, it is convenient to have position r as a function of eccentric anomaly.
- This is obtained by substituting equation:

$$\cos\theta = \frac{\cosh F - e}{1 - e\cosh F}$$

- Into equation

$$r = a\frac{e^2 - 1}{1 - e\cosh F}$$

- This reduces to:

$$r = a(e\cosh F - 1) \qquad (10)$$