Chapter 11

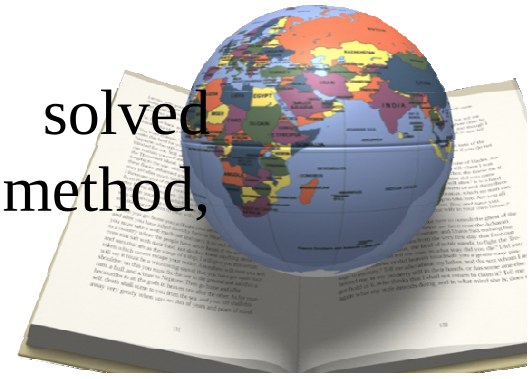
Orbital position as a function of time

CHAPTER CONTENT

11- ORBITAL POSITION AS A FUNCTION OF TIME

11- ORBITAL POSITION AS A FUNCTION OF TIME

- In preview chapter we found the relationship between position and true anomaly for the two-body problem.
- The only place time appeared explicitly was in the expression for the period of an ellipse.
- Obtaining position as a function of time is a simple matter for circular orbits.
- For elliptical, parabolic and hyperbolic paths we are led to the various forms of Kepler's equation relating position to time.
- These transcendental equations must be solved iteratively using a procedure like Newton's method, which is presented in this chapter.



11- ORBITAL POSITION AS A FUNCTION OF TIME

11.1 Time Since Periapsis

- The orbit formula, gives the position of body m_2 in its orbit around m_1 as a function of the true anomaly

$$r = (h^2/\mu)/(1 + e\cos\theta)$$

- **For many practical reasons we need to be able to determine the position of m_2 as a function of time.**
- **For elliptical orbits we have formula for the period T:**

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3$$

- **But we cannot yet calculate the time required to fly between any two anomalies. The purpose of this section is to come up with formulas that allow us to do that calculation**

11- ORBITAL POSITION AS A FUNCTION OF TIME

- The one equation which relates true anomaly directly to time is:

$$h = r^2\dot{\theta} \longrightarrow \frac{d\theta}{dt} = \frac{h}{r^2}$$

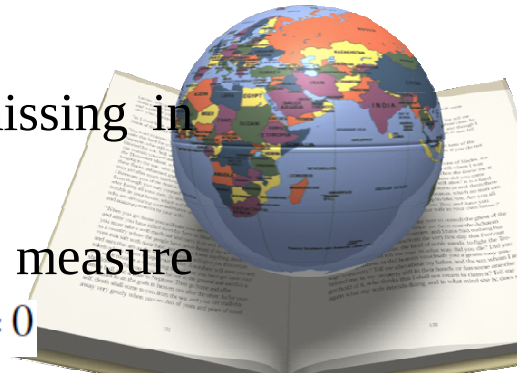
- Substituting r from orbit formula, after separating variables we find:

$$\frac{\mu^2}{h^3}dt = \frac{d\theta}{(1+e\cos\theta)^2}$$

- Integrating both sides of this equation yields:

$$\frac{\mu^2}{h^3}(t-t_p) = \int_0^{\theta} \frac{d\vartheta}{(1+e\cos\vartheta)^2} \qquad (1)$$

- t_p : time at periapse passage ($\theta = 0$)
- t_p is the sixth constant of the motion that was missing in previous chapter.
- The origin of time is arbitrary. It is convenient to measure time from periapse passage so we will usually set $t_p = 0$



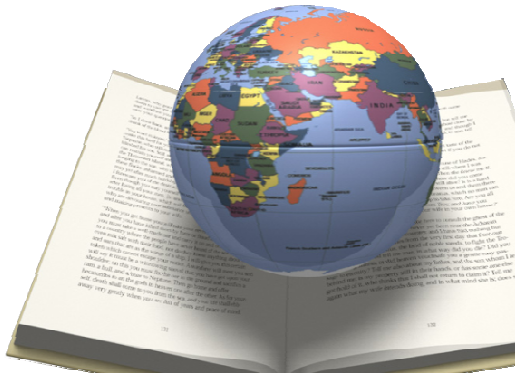
11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ If $t_p = 0$, in that case we have

$$\frac{\mu^2}{h^3} t = \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2} \qquad (2)$$

✶ The integral on the right maybe found in any standard mathematical handbook.

✶ the specific form of the integral depends on whether the value of the eccentricity e corresponds to a circle, ellipse, parabola or hyperbola



11.2 Circular Orbits

★ For a circle, $e = 0$ so the integral in Equation (3) is simply:

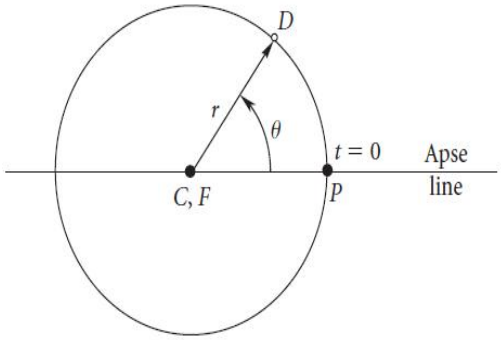
$$\int_0^{\theta} d\vartheta. \longrightarrow t = \frac{h^3}{\mu^2}\theta$$

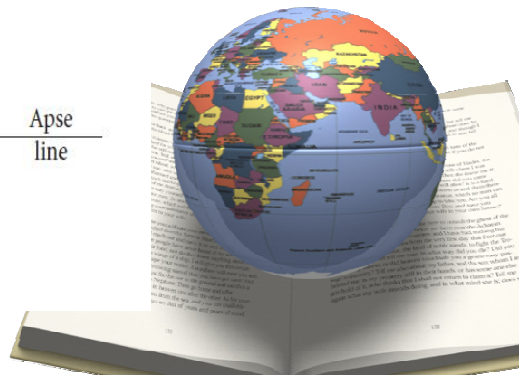
★ Recall that for a circle:

$$r = h^2/\mu \longrightarrow h^3 = r^{\frac{3}{2}}\mu^{\frac{3}{2}} \longrightarrow t = \frac{r^{\frac{3}{2}}}{\sqrt{\mu}}\theta$$

★ Substituting the formula for the period T of a circular orbit yields: (NOTE 18 P(109),{1})

$$\theta = \frac{2\pi}{T}t$$





11.3 Elliptical Orbits

✶ For 0<e<1, we find in integral tables that

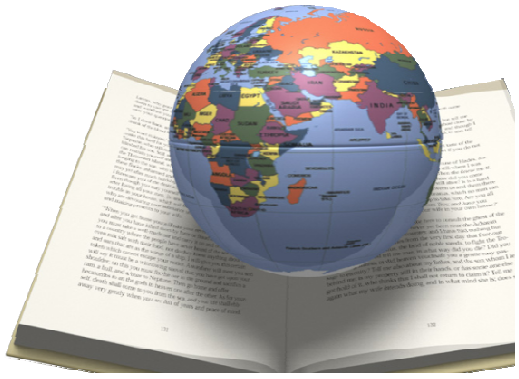
$$\int_0^{\theta} \frac{d\vartheta}{(1+e\cos\vartheta)^2} = \frac{1}{(1-e^2)^{\frac{3}{2}}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

✶ Therefore, Equation (2) in this case becomes:

$$\frac{\mu^2}{h^3}t = \frac{1}{(1-e^2)^{\frac{3}{2}}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

✶ Or

$$M_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \qquad (3)$$

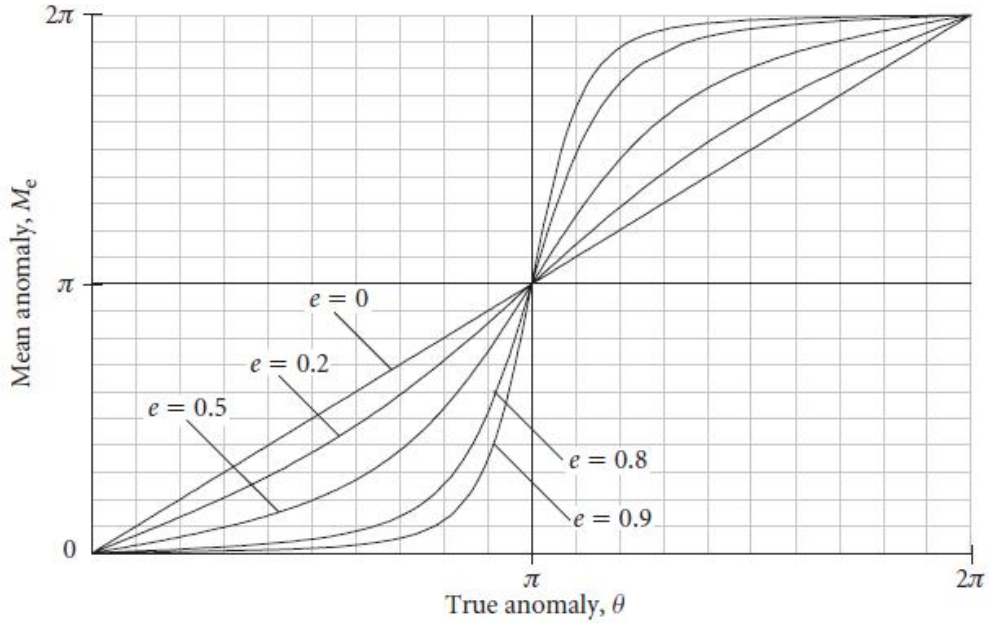


11- ORBITAL POSITION AS A FUNCTION OF TIME

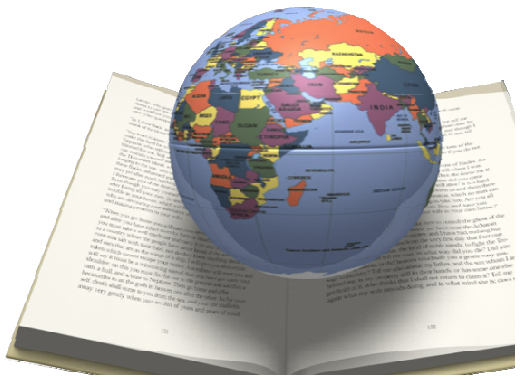
✶ In equation (3), M_e is: called the mean anomaly:

$$M_e = \frac{\mu^2}{h^3}(1 - e^2)^{\frac{3}{2}} t \qquad (4)$$

✶ Equation (3) in plotted in the below figure:



✶ (NOTE 19 PAGE 110, {1})



11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ From the formula for the period T of an elliptical orbit we have:

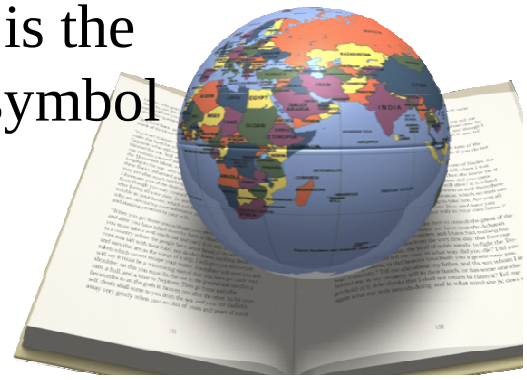
$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 \longrightarrow \mu^2(1-e^2)^{\frac{3}{2}}/h^3 = 2\pi/T$$

✶ So that the mean anomaly can be written much more simply as:

$$M_e = \frac{2\pi}{T}t \qquad (5)$$

✶ The angular velocity of the position vector of an elliptical orbit is not constant, but since 2π radians are swept out per period T, the ratio $2\pi/T$ is the averagee angular velocity which is given the symbol n and called the mean motion.

$$n = \frac{2\pi}{T} \qquad (6)$$



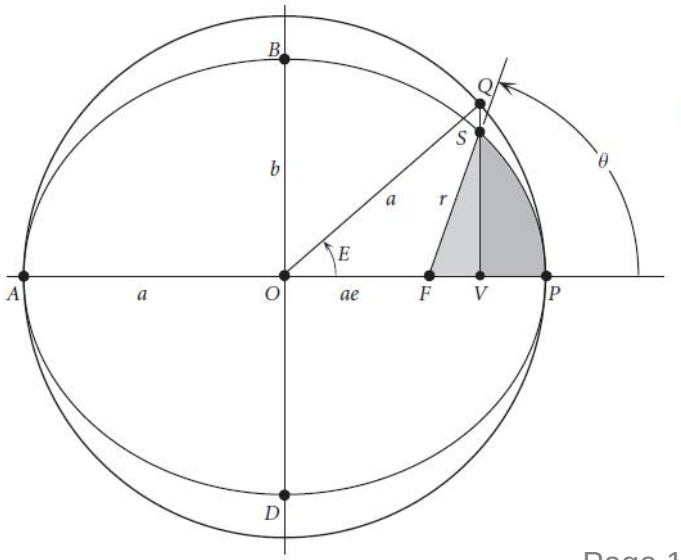
11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ In terms of the mean motion, Equation (5) can be written simpler still:

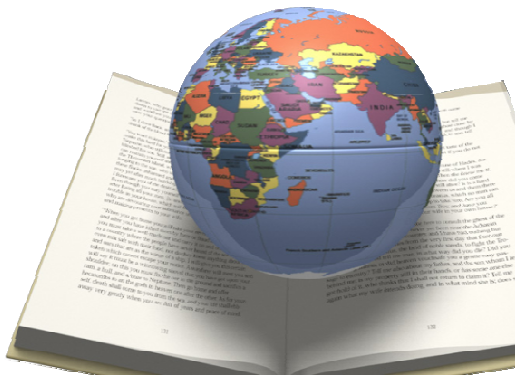
$$M_e = nt$$

✶ (NOTE 20 P 111, {1})

✶ It is convenient to simplify Equation (3), by introducing an auxiliary angle E called the eccentric anomaly. (NOTE 21, P 111, {1})



$$M_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}$$

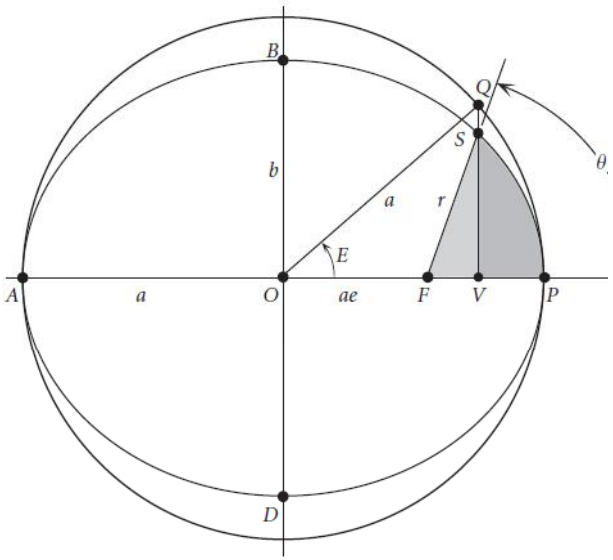


11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ To find E as a function of θ , we first observe from previous figure that:

$$\overline{OV} = a\cos E$$

$$\overline{OV} = ae + r\cos\theta$$



✶ Thus:

$$a\cos E = ae + r\cos\theta$$

✶ Using Equation:

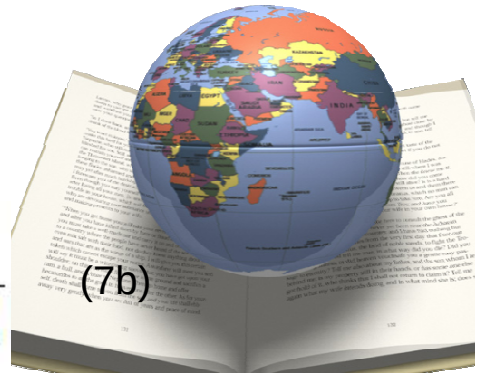
$$r = a(1 - e^2)/(1 + e\cos\theta)$$

✶ We can write this as:

$$a\cos E = ae + \frac{a(1 - e^2)\cos\theta}{1 + e\cos\theta}$$

✶ Simplifying the right-hand side, we get

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} \quad \text{(7a)} \longrightarrow \quad \cos\theta = \frac{e - \cos E}{e\cos E - 1}$$



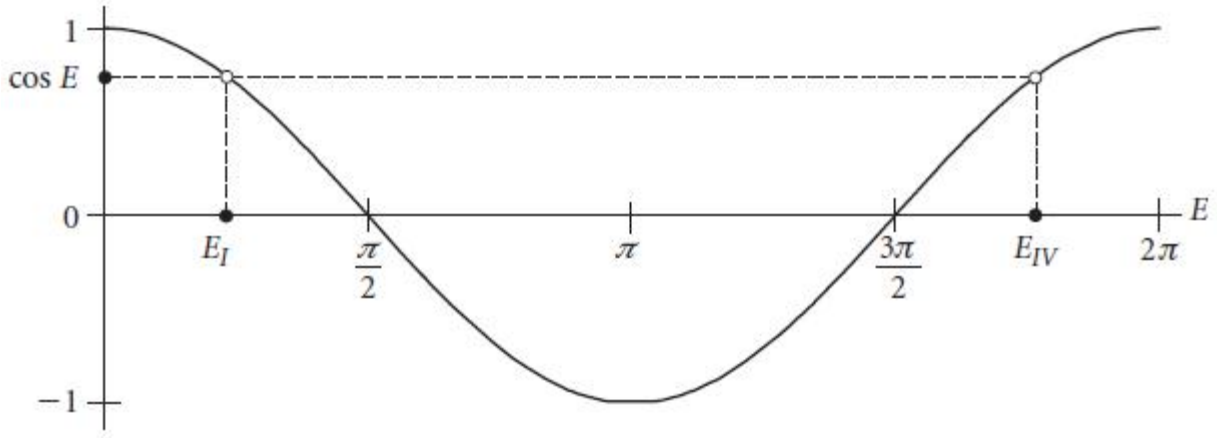
(7b)

11- ORBITAL POSITION AS A FUNCTION OF TIME

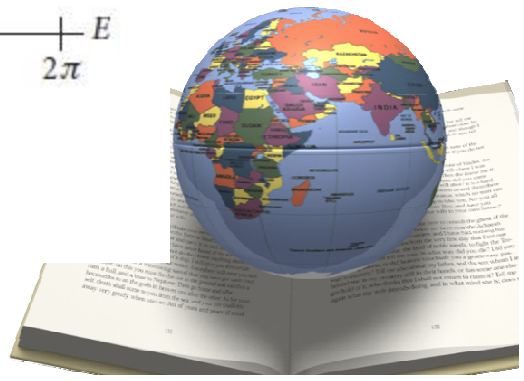
- ✶ Substituting Equation(7a) into the trigonometric identity($\sin^2 E + \cos^2 E = 1$) and solving for $\sin E$
- ✶ yields:

$$\sin E = \frac{\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta} \qquad (8)$$

- ✶ Equation (7a) would be find for obtaining E from θ except that, given a value of $\cos E$ between -1 and 1, there are two values of E between $0°$ and $360°$



- ✶ The same comments hold for Equation(8)



11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ To resolve this quadrant ambiguity, we use the following trigonometric identitiy:

$$\tan^2 \frac{E}{2} = \frac{1 - \cos E}{1 + \cos E} \qquad (9)$$

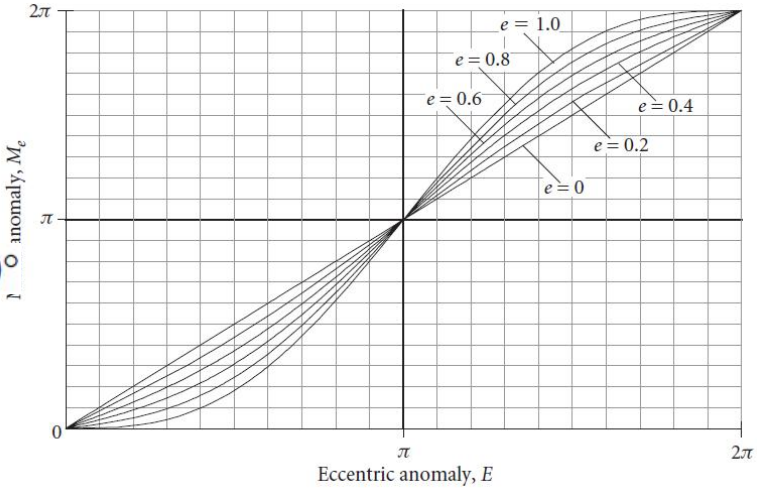
✶ By the use of Equation's (9) and (7a), we obtain:

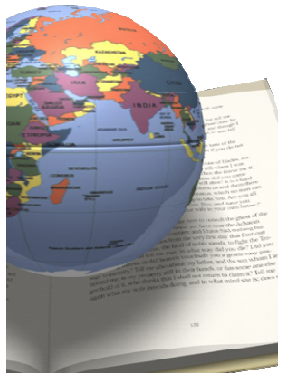
$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2} \qquad (10a)$$

✶ Or

$$E = 2 \tan^{-1} \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2} \right) \qquad (10b)$$

✶ Observe from the above figure that for any value of $\tan(E/2)$, there is only one value of E between $0°$ and $360°$ there is no quadrant ambiguity.



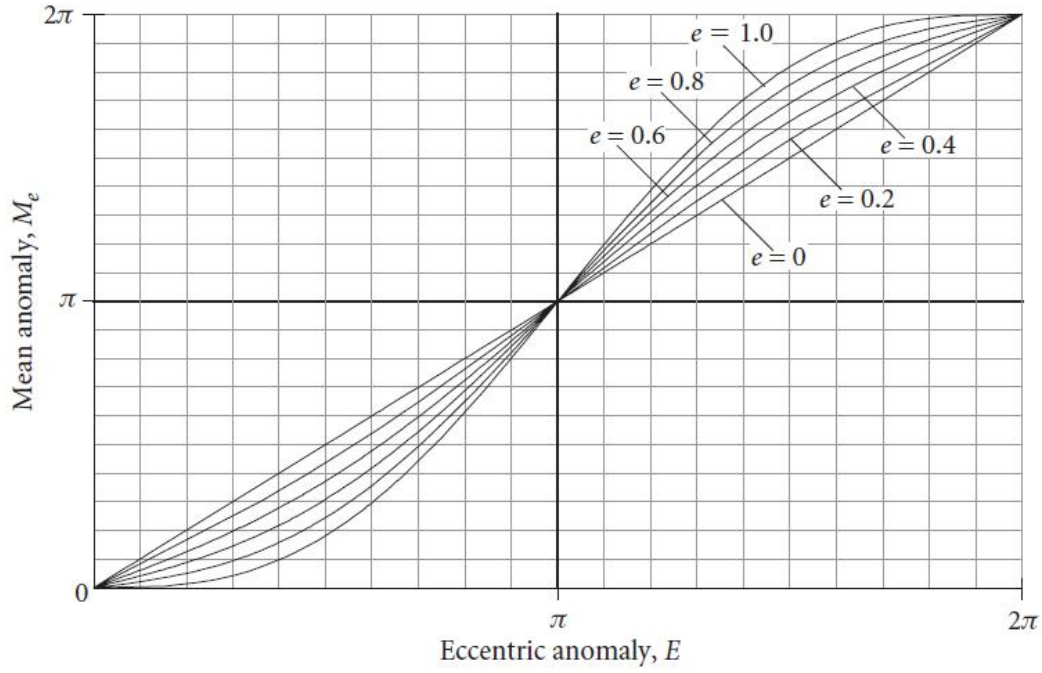


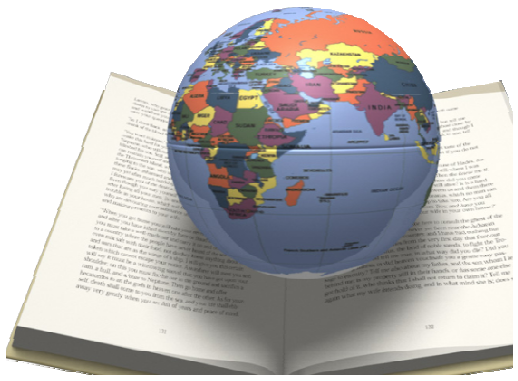
11- ORBITAL POSITION AS A FUNCTION OF TIME

✶ Substituting Equation(8) and (10b) into Equation (3) yield's Kepler's equation:

$$M_e = E - e \sin E \quad (11)$$

✶ This monotonically increasing relationship between mean anomaly and eccentric anomaly is plotted for several values of eccentricity.





11- ORBITAL POSITION AS A FUNCTION OF TIME

How determine the time since preapsis?

1-Given the true anomaly θ, we calculate the eccentric anomaly E using Equations(10)

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} \qquad E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

2-Substituting E into Kepler's formula "Equ.(11)" yields the mean anomaly directly.

$$E - e\sin E = M_e \qquad (12)$$

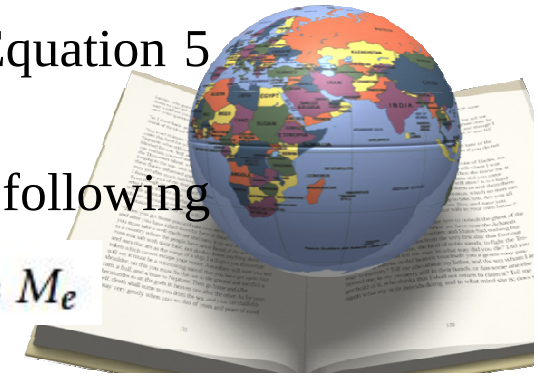
3-From the mean anomaly and the period T we find the time (since periapsis) from Equ:

$$t = \frac{M_e}{2\pi}T$$

✶ On the other hand, if we are given the time, then Equation 5 yields the mean anomaly M_e $\qquad M_e = \frac{2\pi}{T}t$

✶ Substituting M_e into Kepler's equation we get the following expression for the eccentric anomaly.

$$E - e\sin E = M_e$$



11- ORBITAL POSITION AS A FUNCTION OF TIME

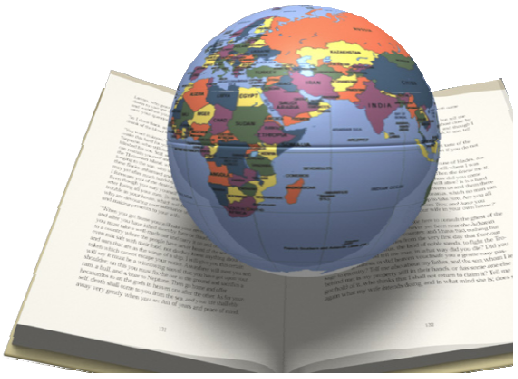
- We cannot solve this transcendental equation directly for E. (A rough value of E might be read of previous figure)
- However, an accurate solution requires an iterative, “trial and error” procedure.
- Newton’s method, or one of its variants, is one of the more common and efficient ways of finding the root of a well-behaved function.

  (NOTE21,P114,{1})
- To apply Newton’s method to the solution of Kepler’s equation, we form the function,

$$f(E) = E - e\sin E - M_e$$

And seek the value of eccentric anomaly that makes

$$f(E) = 0. \text{ Since } f'(E) = 1 - e\cos E$$

- For this problem we have:

$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M_e}{1 - e\cos E_i} \qquad (13)$$