CHAPTER 3

NEWTON’S LAW
OF GRAVITATION

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NEWTON’S LAW OF GRAVITATION
Two men, Tycho Brache and Johan Kepler, laid the groundwork for Newton’s greatest discoveries, 50 years later than his birth. (1592)

Tycho, was recording accurate data on the position of the planets.

Kepler by using the Tycho’s data found and published his three law of planetary motion (1601-1619)

**KEPLER’S LAWS**

First Law—The orbit of each planet is an ellipse, with the sun at a focus.

Second Law—The line joining the planet to the sun sweeps out equal areas in equal times.

Third Law—The square of the period of a planet is proportional to the cube of its mean distance from the sun.
Still, Kepler’s laws were only a description not an explanation of planetary motion.

The 23-year-old Newton conceived the law of gravitation, the laws of motion and developed the fundamental concepts of differential calculus. (1665-1666)

Newton publish his discoveries some 20 years later, in book “The Mathematical Principles of Natural Philosophy” or more simply “The Principia” (1687).

In book I of the principle Newton introduces his three laws of motion:

NEWTON’S LAWS

**First Law**—Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

**Second Law**—The rate of change of momentum is proportional to the force impressed and is in the same direction as that force.

**Third Law**—To every action there is always opposed an equal reaction.
The second law can be expressed mathematically as follows:

\[ \sum F = m \ddot{r} \]  

\( \sum F \): The vector sum of all forces acting on the mass

\( m \ddot{r} \): The vector acceleration of the mass measured relative to an inertial reference from
Newton formulated the law of gravity by stating that any two bodies attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them:

\[ F_g = -\frac{GMm}{r^2} \hat{r} \]  

\( F_g \) : The force on mass m due to mass M  
\( r \) : The vector from M to m  
\( G \): \( 6.6742 \times 10^{11} \text{ m}^3/\text{kg} \cdot \text{s}^2 \) The universal gravitational constant
Mass, like length and time is a primitive physical concept

It can’t be defined in terms of any other physical concept.

Mass is simply the quantity of matter.

More practically, mass is a measure of the inertia of a body.

Inertia in an object’s resistance to changing its state of motion.

The unit of mass is “Kg”
2-NEWTON’S LAW OF GRAVITATION

★ Force is the action of one physical body on another, either through direct contact or through a distance.

★ Gravity is an example of force acting through a distance.

★ The gravitational force between two masses $m_1$ and $m_2$ having a distance $r$ between their centers is:

(Newton’s law of gravity)

$$F_g = G \frac{m_1 m_2}{r^2} \quad (3)$$

$G$: universal gravitational constant

★ The force of gravity is too small unless at least one of the masses is extremely big.
The force of a large mass (such as the earth) on a mass many orders of magnitude smaller (such as a person) is called weight.

The weight of the small body is:

\[ W = G \frac{Mm}{r^2} = m \left( \frac{GM}{r^2} \right) \]

\[ W = mg \]  \hspace{1cm} (4)

\[ g = \frac{GM}{r^2} \]  \hspace{1cm} (5)

\( g \) : (m/s\(^2\)) acceleration of gravity
If planetary gravity is the only force acting on a body the body is said to be in free fall.

In free fall, there are no contact forces, so there can be no sense of weight.

Even though the weight is not zero, a person in free fall experiences weightlessness, or absence of gravity.

$$ R_E = 6378 km \quad \longrightarrow \quad g_0 = \frac{GM}{R_E^2} \quad (6) $$

$$ g_0 = \frac{GM}{R_E^2} \quad \longrightarrow \quad g_0 = 9.807 \, m/s^2 $$
Measurement's show that for altitudes on the order of 10 kilometers, $g$ is only three-tenths of a percent ($0.3\%$) less than its sea-level value.

Thus under ordinary conditions, we ignore the variation of $g$ with altitude.

At space station altitude (300 km), weight is only about 10 percent less than it is on the earth’s surface.
Show that in the absence of an atmosphere, the shape of a low altitude ballistic trajectory is a parabola. Assume the acceleration of gravity $g$ is constant and neglect the earth’s curvature.
Example 2.1

$\dot{t} = 0$: Time of launched
$v_0$: speed
$\gamma_0$: Flight path angle

Solution:

Since the projectile is in free fall after launch, its only acceleration is that of gravity in the negative $y$-direction

$$\ddot{x} = 0$$

$$\ddot{y} = -g$$
Integrating with respect to time and applying the initial conditions leads to:

\[ x = x_0 + (v_0 \cos \gamma_0) t \]  \hspace{1cm} (a)

\[ y = y_0 + (v_0 \sin \gamma_0) t - \frac{1}{2} gt^2 \]  \hspace{1cm} (b)

Solving (a) for t and substituting the result into (b) yields:

\[ y = y_0 + (x - x_0) \tan \gamma_0 - \frac{1}{2} \frac{g}{v_0 \cos \gamma_0} (x - x_0)^2 \]
An airplane flies a parabolic trajectory so that the passengers will experience free fall (weightlessness).

What is the required variation of the flight path angle $\gamma$ with speed $\nu$?
Solution:

ɨ For “flat” earth \( d\gamma = -d\phi \rightarrow \dot{\gamma} = -\phi \)

ɨ We have had: \( \rho \dot{\gamma} = v \) \tag{g} \n
ɨ The normal acceleration \( a_n \) is just the component of the gravitational acceleration \( g \) then:
\[
a_n = g \cos \gamma \quad \tag{a}
\]
Substituting \( a_n = \frac{v^2}{\rho} \) into (a) and solving for the radius of curvature yields:

\[
\rho = \frac{v^2}{g \cos \gamma}
\]

Combining equations (g) and (b) we find:

\[
\dot{\gamma} = -\frac{g \cos \gamma}{v}
\]
2-NEWTON’S LAW OF GRAVITATION

★ Force is not primitive concept like mass because it is connected with the concepts of motion and inertia

★ The only way to alter the motion of a body is to exert a force on it.

★ If the resultant or net force on a body of mass \( m \) is,

\[
F_{\text{net}} = ma
\]

(10)
The integral of a force $F$ over a time interval is called the impulse $I$ of the force.

$$I = \int_{t_1}^{t_2} F \, dt \quad (11)$$

$$(10), (11) \quad I_{\text{net}} = \int_{t_1}^{t_2} m \frac{dv}{dt} \, dt = mv_2 - mv_1 \quad (12)$$

That is the net impulse on a body yields a change in its linear momentum so that:

$$\Delta v = \frac{I_{\text{net}}}{m} \quad (13)$$

If $F_{\text{net}}$ is constant, then $I_{\text{net}} = F_{\text{net}} \Delta t$

So, equation (13) becomes:

$$\Delta v = \frac{F_{\text{net}}}{m} \Delta t \quad (14)$$
2-NEWTON’S LAW OF GRAVITATION

★ The moment of the net force about $O$ is

$$M_{O_{\text{net}}} = \mathbf{r} \times \mathbf{F}_{\text{net}}$$

$$M_{O_{\text{net}}} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt}$$

★ If the mass $m$ is constant:

$$\mathbf{r} \times m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) - \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) - (\mathbf{v} \times m\mathbf{v})$$
2-NEWTON’S LAW OF GRAVITATION

★ Since \( \mathbf{v} \times m\mathbf{v} = m(\mathbf{v} \times \mathbf{v}) = 0 \), so

\[
(15) \quad M_{O_{\text{net}}} = \frac{dH_O}{dt} \quad (16)
\]

★ Where \( H_o \) is the angular momentum about \( O \)

\[
H_O = r \times m\mathbf{v} \quad (17)
\]

★ Thus, just as the net force on a particle changes its linear momentum \( m\mathbf{v} \), the moment of that force about a fixed point changes the moment of its linear momentum about that point.

\[
\int_{t_1}^{t_2} M_{O_{\text{net}}} dt = H_{O_2} - H_{O_1} \quad (18)
\]

★ The integral on the left is the net angular impulse
EXAMPLE 2.3

A particle of mass \( m \) is attached to point \( O \) by an inextensible string of length \( l \). Initially the string is slack when \( m \) is moving to the left with a speed \( v_0 \) in the position shown. Calculate the speed of \( m \) just after the string becomes taut. Also, compute the average force in the string over the small time interval \( \Delta t \) required to change the direction of the particle’s motion.
2-NEWTON’S LAW OF GRAVITATION

EXAMPLE 2.3

Initially, the position and velocity of the particle are
\[ \mathbf{r}_1 = c\mathbf{i} + d\mathbf{j} \quad \mathbf{v}_1 = -v_0\mathbf{i} \]

The angular momentum is
\[ \mathbf{H}_1 = \mathbf{r}_1 \times m\mathbf{v}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & d & 0 \\ -mv_0 & 0 & 0 \end{vmatrix} = mv_0\mathbf{k} \quad (a) \]

Just after the string becomes taut
\[ \mathbf{r}_2 = -\sqrt{l^2 - d^2}\mathbf{i} + d\mathbf{j} \quad \mathbf{v}_2 = v_x\mathbf{i} + v_y\mathbf{j} \quad (b) \]

And the angular momentum is
\[ \mathbf{H}_2 = \mathbf{r}_2 \times m\mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{l^2 - d^2} & d & 0 \\ v_x & v_y & 0 \end{vmatrix} = \left(-mv_xd - mv_y\sqrt{l^2 - d^2}\right)\mathbf{k} \quad (c) \]

Initially the force exerted on \( m \) by the slack string is zero. When the string becomes taut, the force exerted on \( m \) passes through \( O \), therefore, the moment of the net force on \( m \) about \( O \) remains zero. \( \mathbf{H}_2 = \mathbf{H}_1 \)
Substituting (a) and (c) yields

\[ v_x d + \sqrt{l^2 - d^2} v_y = -v_0 d \]  \hspace{1cm} (d)

The string is inextensible, so the component of the velocity of m along the string must be zero

\[ \mathbf{v}_2 \cdot \mathbf{r}_2 = 0 \]

Substituting \( v_2 \) and \( r_2 \) from (b) and solving for \( V_y \) we get

\[ v_y = v_x \sqrt{\frac{l^2}{d^2} - 1} \]  \hspace{1cm} (e)

Solving (d) and (e) for \( v_x \) and \( v_y \) leads to

\[ v_x = -\frac{d^2}{l^2} v_0 \quad v_y = -\sqrt{1 - \frac{d^2}{l^2}} \frac{d}{l} v_0 \]  \hspace{1cm} (f)

Thus, the speed, \( v = \sqrt{v_x^2 + v_y^2} \), after the string becomes taut is

\[ v = \frac{d}{l} v_0 \]
From equation 12, the impulse on \( m \) during the time it takes the string become taut is

\[
I = m(v_2 - v_1) = m \left[ \left( -\frac{d^2}{l^2}v_0\hat{i} - \sqrt{1 - \frac{d^2}{l^2}}\frac{d}{l}v_0\hat{j} \right) - (-v_0\hat{i}) \right]
\]

\[
= \left( 1 - \frac{d^2}{l^2} \right) mv_0\hat{i} - \sqrt{1 - \frac{d^2}{l^2}} \frac{d}{l}mv_0\hat{j}
\]

The magnitude of this impulse, which is directed along the string, is

\[
I = \sqrt{1 - \frac{d^2}{l^2}} \frac{1}{2}mv_0
\]

Hence, the average force in the string during the small time interval \( \Delta t \) required to change the direction of the velocity vector turns out to be

\[
F_{avg} = \frac{I}{\Delta t} = \sqrt{1 - \frac{d^2}{l^2}} \frac{mv_0}{\Delta t}
\]