CHAPTER 4

THE N-BODY PROBLEM

CHAPTER CONTENT

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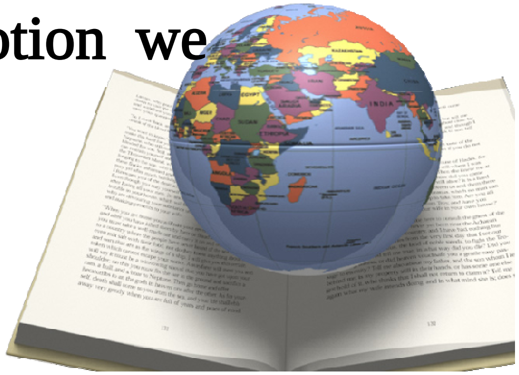
3-THE N-BODY PROBLEM

✶In this section we shall explain the motion of a body which is being acted upon by several gravitational masses and may even be experiencing other forces such as drag, thrust and solar radiation pressure.

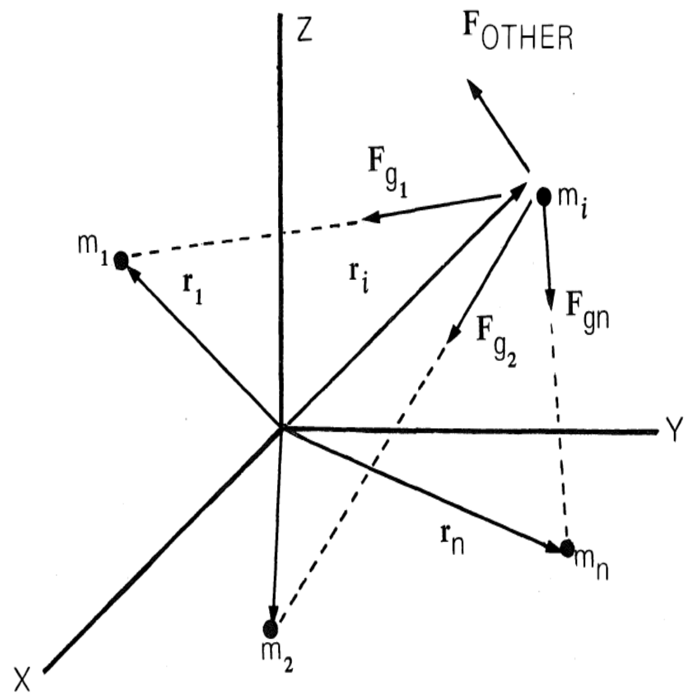
✶ **For this we shall assume a "system" of n-bodies $(m_1, m_2, m_3, \ldots, m_n)$**

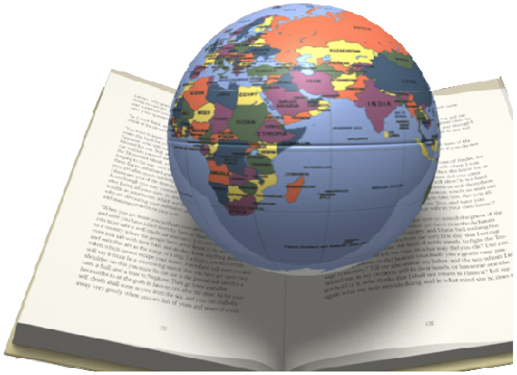
✶**One of these bodies is the body whose motion we wish to study-call it the i^{th} body, m_i**



3-THE N-BODY PROBLEM

- The vector sum of all gravitational forces and other external forces acting on m_i will be used to determine the equation of motion
- To determine the gravitational forces we shall apply Newton's law of universal gravitation. (Note1,page5,{2})





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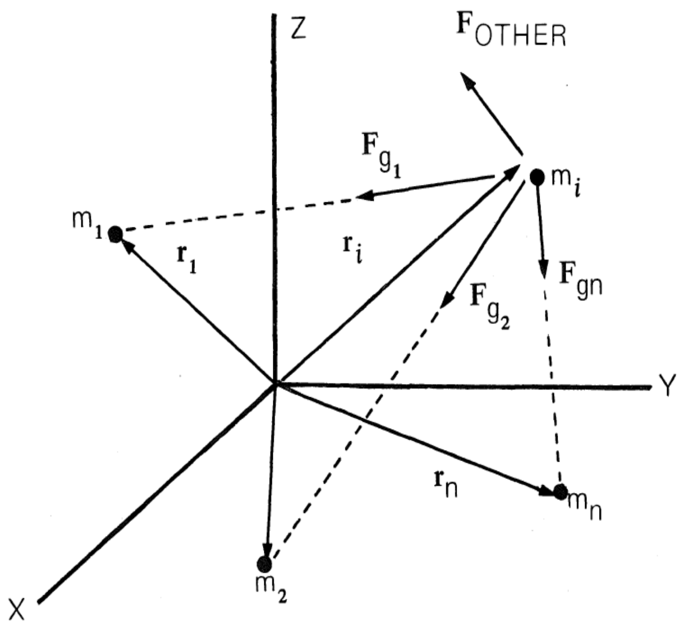
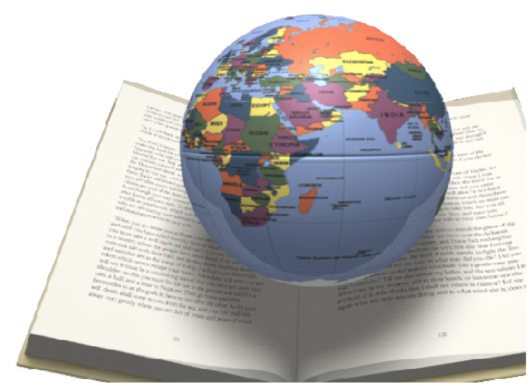
- The first step in our analysis will be to choose a "suitable" coordinate system. This system is illustrated below:



- **In (X,Y,Z) coordinate system, the position of the *n* masses are known**

$$(r_1, r_2, \ldots, r_n)$$



3-THE N-BODY PROBLEM

✶The force F_{gn} exerted on m_i by m_n is: (Newton's law of universal gravitation)

$$\mathbf{F}_{gn} = -\frac{Gm_i m_n}{r_{ni}^3}(\mathbf{r}_{ni}) \qquad (1)$$

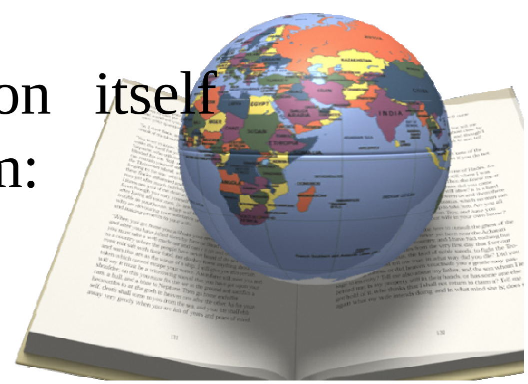
$$\mathbf{r}_{ni} = \mathbf{r}_i - \mathbf{r}_n \qquad (2)$$

✶ The vector sum, F_g , of all gravitational forces acting on the i^{th} body may be written:

$$\mathbf{F}g = -\frac{Gm_i m_1}{r_{1i}^3}(\mathbf{r}_{1i}) - \frac{Gm_i m_2}{r_{2i}^3}(\mathbf{r}_{2i}) \cdots\cdots - \frac{Gm_i m_n}{r_{ni}^3}(\mathbf{r}_{ni}) \qquad (3)$$

✶ Since the body cannot exert a force on itself obviously, equation (3) does not contain the term:

$$-\frac{Gm_i m_i}{r_{ii}^3}(\mathbf{r}_{ii}) \qquad (4)$$



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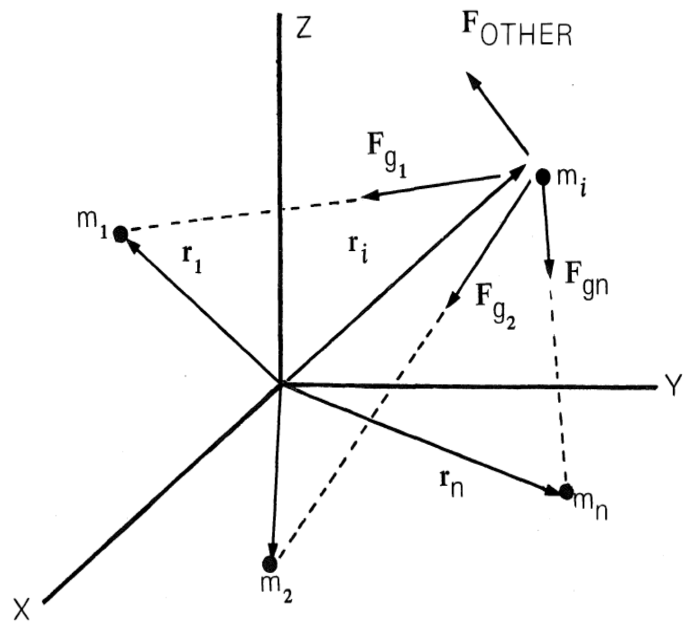
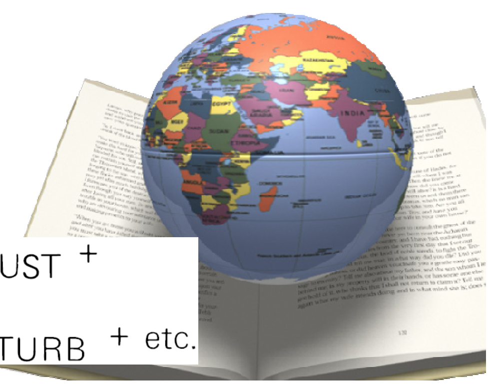
- ✶ We may simplify the equation No3 by using the summation notation so that

$$\mathbf{F}_g = -Gm_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3} (\mathbf{r}_{ji}) \quad \textbf{(5)}$$



- ✶ **The other external force, F_{OTHER}, is composed of drag, thrust, solar radiation pressure, perturbations due to nonspherical shapes, etc.**

$$\mathbf{F}_{OTHER} = \mathbf{F}_{DRAG} + \mathbf{F}_{THRUST} + \mathbf{F}_{SOLAR\ PRESSURE} + \mathbf{F}_{PERTURB} + \text{etc.}$$



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- The combined force acting on the i^{th} body we will call F_{TOTAL}

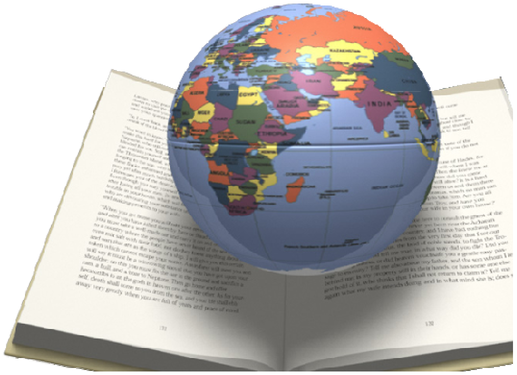
$$\mathbf{F}_{\mathrm{TOTAL}} = \mathbf{F}_{\mathrm{g}} + \mathbf{F}_{\mathrm{OTHER}} \,. \quad (6)$$

- By applying the Newton's second law of motion, we will have

$$\frac{d}{dt}(m_i \mathbf{v}_i) = \mathbf{F}_{\mathrm{TOTAL}} . \quad (7)$$

- The time derivative may be expanded to:

$$m_i \frac{d\mathbf{v}_i}{dt} + \mathbf{v}_i \frac{dm_i}{dt} = \mathbf{F}_{\mathrm{TOTAL}} \,. \quad (8)$$



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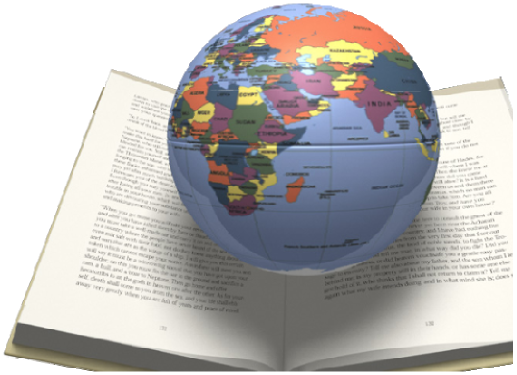
- If the body is expelling some mass, (for example to produce thrust) the second term of equation(8) would not be zero.
- Certain relativistic effects would also give rise to changes in the mass m_i as a function of time.
- **In other words, in space dynamics, it is not true that F=ma.**
- **Dividing through by the mass m_i gives the most general equation of motion for the i^{th} body**

$$\ddot{\mathbf{r}}_i = \frac{\mathbf{F}_{TOTAL}}{m_i} - \dot{\mathbf{r}}_i \frac{\dot{m}_i}{m_i} \qquad \textbf{(9)}$$



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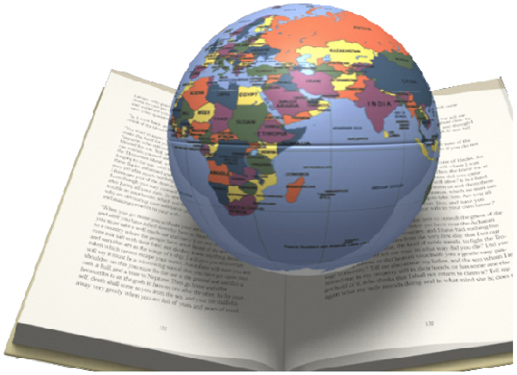
$\ddot{r}_i$: The vector acceleration of the i^{th} body relative to the x,y,z coordinate system.

m_i : The mass of i^{th} body

F_{TOTAL} : The vector sum of all gravitational forces and all other external forces.

$\dot{r}_i$: The velocity vector of the i^{th} body relative to the x,y,z coordinate system.

$\dot{m}_i$: The time rate of change of mass of the i^{th} body (due to expelling mass or relativistic effects)



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$$\ddot{\mathbf{r}}_i = \frac{\mathbf{F}_{TOTAL}}{m_i} - \dot{\mathbf{r}}_i \frac{\dot{m}_i}{m_i}$$

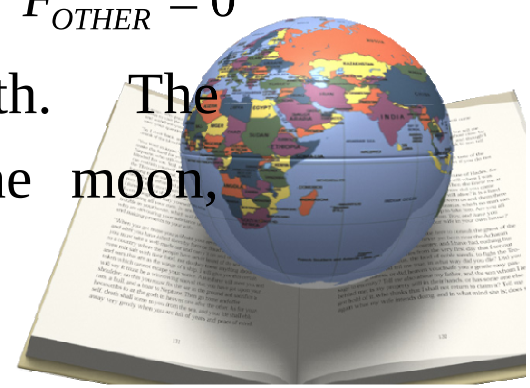
- Equation (9) is a second order, nonlinear vector, differential equation of motion which has defied solution in its present form.

- So we make some simplifying assumptions:

  1- The mass of the i^{th} body remains constant (i.e., unpowered flight $\dot{m}_i = 0$)

  2- The all other external forces are not present $F_{OTHER} = 0$

  3- m_2 is an earth satellite and m_1 is the earth. The remaining masses $m_3, m_4, \ldots, m_n$ may be the moon, sun and planets.



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- From the first 2 assumptions we will write equation 9 in the following form:

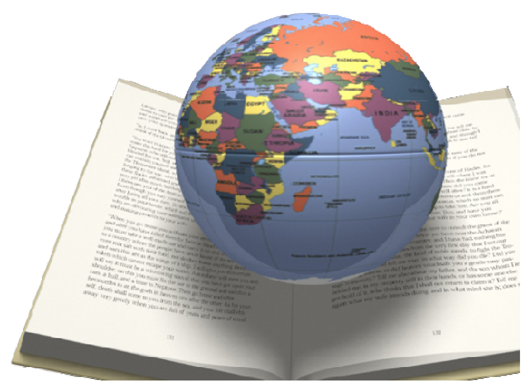
$$\ddot{\mathbf{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^{3}} (\bar{\mathbf{r}}_{ji}) . \tag{10}$$

- By using the 3 assumption for i=1 we will have

$$\ddot{\mathbf{r}}_1 = -G \sum_{j=2}^{n} \frac{m_j}{r_{j1}^{3}} (\mathbf{r}_{j1}) . \tag{11}$$

- And for i=2 equation 10 becomes

$$\ddot{\mathbf{r}}_2 = -G \sum_{\substack{j=1 \\ j \neq 2}}^{n} \frac{m_j}{r_{j2}^{3}} (\mathbf{r}_{j2}) . \tag{12}$$



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✶ From equation 2 we see that:

$$\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \qquad (13)$$

✶ So that:

$$\ddot{\mathbf{r}}_{12} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 \qquad (14)$$

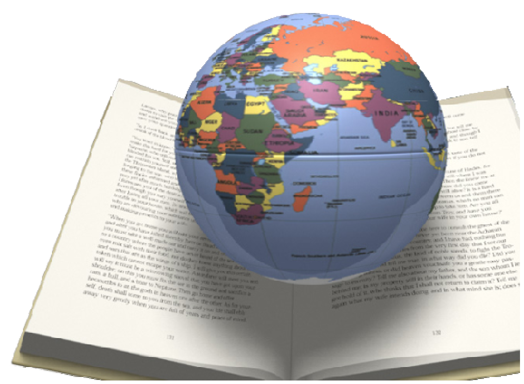
✶ Substituting equations (11) and (12) into equation (14) gives:

$$\ddot{\mathbf{r}}_{12} = -G \sum_{\substack{j=1 \\ i \neq 2}}^{n} \frac{m_j}{r_{j2}^{3}} (\mathbf{r}_{j2}) + G \sum_{j=2}^{n} \frac{m_j}{r_{j1}^{3}} (\mathbf{r}_{j1}) \qquad (15)$$

✶ Or expanding

$$\ddot{\mathbf{r}}_{12} = -\left[\frac{Gm_1}{r_{12}^{3}} (\mathbf{r}_{12}) + G \sum_{j=3}^{n} \frac{m_j}{r_{j2}^{3}} (\mathbf{r}_{j2}) \right]$$

$$-\left[-\frac{Gm_2}{r_{21}^{3}} (\mathbf{r}_{21}) - G \sum_{j=3}^{n} \frac{m_j}{r_{j1}^{3}} (\mathbf{r}_{j1}) \right] \qquad (16)$$



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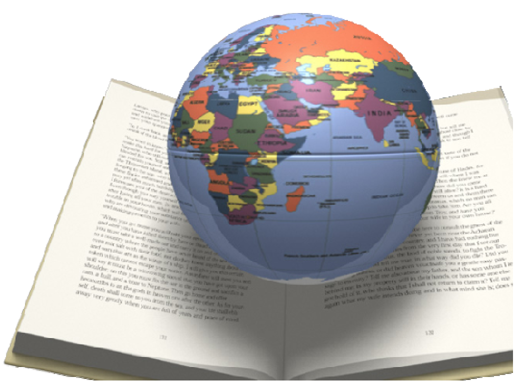
- Since $\mathbf{r}_{12} = -\mathbf{r}_{21}$ we may combine the first terms in each bracket. Hence:

$$\ddot{\mathbf{r}}_{12} = -\frac{G(m_1 + m_2)}{r_{12}^{3}}(\mathbf{r}_{12}) - \sum_{j=3}^{n} Gm_j\left(\frac{\mathbf{r}_{j2}}{r_{j2}^{3}} - \frac{\mathbf{r}_{j1}}{r_{j1}^{3}}\right) \qquad \textbf{(17)}$$



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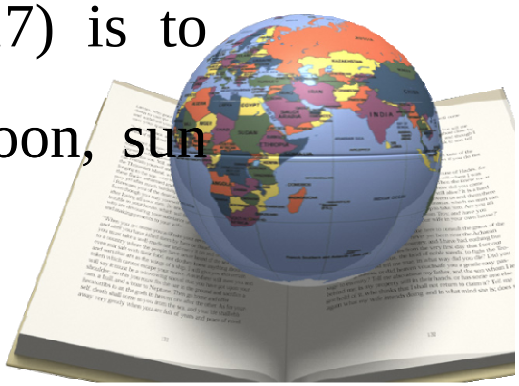
- If we are going to study the motion of a near earth satellite, so we could assume that, m_2 is the mass of the satellite and m_1 is the mass of the earth. In equation (17).
- Then from equation (17) $\ddot{r}_{12}$ is the acceleration of the satellite relative to earth.
- The effect of the last term of equation (17) is to account for the perturbing effects of the moon, sun and planets on a near earth satellite.



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- To further simplify this equation it is necessary to determine the magnitude of the perturbing effects compared to the force between earth and satellite. (note no2 , page11 {2})

COMPARISON OF RELATIVE ACCELERATION (IN G's) FOR A 200 NM EARTH SATELLITE

	Acceleration in G's on 200 nm Earth Satellite
Earth	.89
Sun	$6x10^{-4}$
Mercury	$2.6x10^{-10}$
Venus	$1.9x10^{-8}$
Mars	$7.1x10^{-10}$
Jupiter	$3.2x10^{-8}$
Saturn	$2.3x10^{-9}$
Uranus	$8x10^{-11}$
Neptune	$3.6x10^{-11}$
Pluto	10^{-12}
Moon	$3.3x10^{-6}$
Earth Oblateness	10^{-3}