CHAPTER 10

Hyperbolic Trajectories

(e > 1)

CHAPTER CONTENT
10- HYPERBOLIC TRAJECTORIES (e > 1)
If $e > 1$, the orbit formula describes the geometry of the hyperbola

$$r = \frac{h^2}{\mu} \frac{1}{1 + e \cos \theta} \quad (1)$$

★ The system consists of two symmetric curves
★ One of the occupied by the orbiting body, the other one is its empty, mathematical image
10- HYPERBOLIC TRAJECTORIES ($e > 1$)

★ Clearly: \[ \lim r \to \& \cos \theta \to -\frac{1}{e} \]

★ We denote this value of true anomaly since the radial distance approaches infinity as the true anomaly approaches $\theta_\infty$.

\[ \theta_\infty = \cos^{-1}(-1/e) \quad (2) \]

★ $\theta_\infty$ is known as the true of the asymptote.

★ Observe that $\theta_\infty$ lies between $90^\circ$ and $180^\circ$

★ From trigonometry it follow that \[ \sin \theta_\infty = \frac{\sqrt{e^2 - 1}}{e} \quad (3) \]
10- HYPERBOLIC TRAJECTORIES (\(e >1\))

- For \(-\theta_\infty < \theta < \theta_\infty\), the physical trajectory is the occupied hyperbola I (on the left).
- For \(\theta_\infty < \theta < (360^\circ - \theta_\infty)\), hyperbola II - the vacant orbit around the empty focus \(F'\) - is traced out. (NOTE17,P69,\{1\})
- Periapsis P lies on the apse line on the physical hyperbola I, whereas apoapsis A lies on the apse line on the vacant orbit.
- The point halfway between periapsis and apoapsis is the center C of the hyperbola.
10- HYPERBOLIC TRAJECTORIES ($e > 1$)

- The asymptotes intersect at C, making angle $\beta$ with the apse line.
  \[ \beta = 180^\circ - \theta_\infty. \]
  \[ \cos \beta = -\cos \theta_\infty. \]
  \[ \beta = \cos^{-1}(1/e) \] (2)
- The angle $\delta$ between the asymptotes is called the turn angle.
- The turn angle is the angle through which the velocity vector of the orbiting body is rotated as it rounds the attracting body at F and heads back towards infinity.

\[ \delta = 180^\circ - 2\beta, \quad \sin \frac{\delta}{2} = \sin \left( \frac{180^\circ - 2\beta}{2} \right) = \sin(90^\circ - \beta) = \cos \beta \quad \Rightarrow \quad \delta = 2 \sin^{-1}(1/e) \]
The distance \( r_p \) from the focus \( F \) to the periapsis is given by equation:

\[
r_a = \frac{h^2}{\mu \left( 1 + e \right)}
\]  

(6)

The radial coordinate \( r_a \) of apoapsis is found by setting \( \theta = 180^\circ \) in equation:

\[
r = \frac{h^2}{\mu \left( 1 + e \cos \theta \right)}
\]  

(7)

so

\[
r_a = \frac{h^2}{\mu \left( 1 - e \right)}
\]

Observe that \( r_a \) is negative, since \( e > 1 \) for the hyperbola. That means the apoapsis lies to the right of the focus \( F \).
We see that the distance $2a$ from periapse $P$ to apoapse $A$ is:

$$2a = |r_a| - r_p = -r_a - r_p$$

Substituting equation (6), (7) yields

$$2a = -\frac{h^2}{\mu} \left( \frac{1}{1-e} + \frac{1}{1+e} \right) \quad a = \frac{h^2}{\mu} \frac{1}{e^2 - 1} \quad (8)$$

So the orbit formula may be written for the hyperbola

$$r = a \frac{e^2 - 1}{1 + e \cos \theta} \quad (9)$$
From equation (g) it follows that:

\[ r_p = a(e - 1) \quad (10) \]

\[ r_a = -a(e + 1) \quad (11) \]

The distance \( b \), from periapsis to an asymptote measured perpendicular to the apse line; is the semiminor axis of the hyperbola.

The length \( b \) is

\[ b = a \tan \beta = a \frac{\sin \beta}{\cos \beta} = a \frac{\sin (180 - \theta_\infty)}{\cos (180 - \theta_\infty)} = a \frac{\sin \theta_\infty}{-\cos \theta_\infty} = a \frac{\sqrt{e^2 - 1}}{e} \left( -\frac{1}{e} \right) \quad (12) \]
The distance \( \Delta \) between the asymptote and a parallel line through the focus is called the aiming radius.

We see that

\[ \Delta = (r_p + a) \sin \beta \]

(10) \[ \Delta = ae \sin \beta \]

(4) \[ \Delta = ae \frac{\sqrt{e^2 - 1}}{e} \]

(3) \[ \Delta = ae \sin \theta_\infty = ae \sqrt{1 - \cos^2 \theta_\infty} \]

(2) \[ \Delta = ae \sqrt{1 - \frac{1}{e^2}} \]
Finally:  \[ \Delta = a \sqrt{e^2 - 1} \]  \hspace{1cm} (13)

Comparing this result with equation 12, it is clear that the aiming radius equals the length of the semiminor axis of the hyperbola.

As with the ellipse and the parabola, we can express the polar form of the equation of the hyperbola in a cartesian coordinate system whose origin is in this case midway between the two foci.
From the figure it is apparent that:

\[ x = -a - r_p + r \cos \theta \]  \hspace{1cm} (14)

\[ y = r \sin \theta \]  \hspace{1cm} (15)

Using equation (9), (10), (14) we obtain:

\[ x = -a - a(e - 1) + a \frac{e^2 - 1}{1 + e \cos \theta} \cos \theta = -a \frac{e + \cos \theta}{1 + e \cos \theta} \]

Substituting equation (9) and (12) in (15) we obtain:

\[ y = \frac{b}{\sqrt{e^2 - 1}} \frac{e^2 - 1}{1 + e \cos \theta} \sin \theta = b \frac{\sqrt{e^2 - 1} \sin \theta}{1 + e \cos \theta} \]
It follows that:

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left( \frac{e + \cos \theta}{1 + e \cos \theta} \right)^2 - \left( \frac{\sqrt{e^2 - 1} \sin \theta}{1 + e \cos \theta} \right)^2 \\
= \frac{e^2 + 2e \cos \theta + \cos^2 \theta - (e^2 - 1)(1 - \cos^2 \theta)}{(1 + e \cos \theta)^2} \\
= \frac{1 + 2e \cos \theta + e^2 \cos^2 \theta}{(1 + e \cos \theta)^2} = \frac{(1 + e \cos \theta)^2}{(1 + e \cos \theta)^2}
\]

That is,

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 
\]

this is the familiar equation of hyperbola which is symmetric about x and y axes, with intercept on the x axis.
10- HYPERBOLIC TRAJECTORIES (e > 1)

★ The specific energy of the hyperbolic trajectory is:

$$\varepsilon = -\frac{1}{2} \frac{\mu^2}{h^2} (1 - e^2)$$

$$a = \frac{h^2}{\mu} \frac{1}{e^2 - 1}$$

$$\varepsilon = \frac{\mu}{2a}$$  \hspace{1cm} (17)

★ The specific energy of a hyperbolic orbit is clearly positive and independent of the eccentricity.

★ The conservation of energy for a hyperbolic trajectory is:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$  \hspace{1cm} (18)

★ Let \(v_\infty\) denote the speed at which a body on a hyperbolic path arrives at infinity so:

$$v_\infty = \sqrt{\frac{\mu}{a}}$$  \hspace{1cm} (19)
In terms of \( v_\infty \) we may write equation (18) as:

\[
\frac{v^2}{2} - \frac{\mu}{r} = \frac{v^2_\infty}{2}
\]

\( v_\infty \) is called the hyperbolic excess speed.

Substituting the expression for escape speed, we obtain for a hyperbolic trajectory

\[
v^2 = v^2_{\text{esc}} + v^2_\infty \quad \text{(19)}
\]

This equation clearly shows that the hyperbolic excess speed \( v_\infty \) represent the excess kinetic energy over that which is required to simply escape from the center of attraction.
The square of \( v_\infty \) is denoted \( C_3 \), and is known as the characteristic energy

\[
C_3 = v_\infty^2 \quad (20)
\]

\( C_3 \) is a measure of the energy required for an interplanetary mission and \( C_3 \) is also a measure of maximum energy a launch vehicle can import to a spacecraft of a given mass

\[
C_3)_{\text{launch vehicle}} > C_3)_{\text{mission}}
\]

\( v_\infty \) can be find also:

\[
v_\infty = \frac{\mu}{h} e \sin \theta_\infty = \frac{\mu}{h} \sqrt{e^2 - 1} \quad (21)
\]
The figure shows a range of trajectories, from a circle through hyperbolas, all having common focus and periapsis.
EXAMPLE 10-1

At a given point of a spacecraft’s geocentric trajectory, the radius is 14600km, the speed is 8.6km/s, and the flight path angle is 50°. Show that the path is a hyperbola and calculate the following: (a) $C_3$, (b) angular momentum, (c) true anomaly, (d) eccentricity, (e) radius of perigee, (f) turn angle, (g) semimajor axis, and (h) aiming radius.

To determine the type of the trajectory, calculate the escape speed at the given radius.

\[
\nu_{\text{esc}} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2 \cdot 398600}{14600}} = 7.389 \text{ km/s}
\]

Since the escape speed is less than the spacecraft’s speed of 8.6km/s, the path is a hyperbola.
10- HYPERBOLIC TRAJECTORIES (\( \varepsilon > 1 \))

(a) the hyperbolic excess velocity \( v_\infty \) is found from equation (19),

\[
v_\infty^2 = v^2 - v_{esc}^2 = 8.6^2 - 7.389^2 = 19.36 \text{ km}^2/\text{s}^2
\]

From equation (20) it follows that

\[
C_3 = 19.36 \text{ km}^2/\text{s}^2
\]

(b) Knowing the speed and the flight path angle, we can obtain

**both** \( v_r \) **and** \( v_\perp \):

\[
v_r = v \sin \gamma = 8.6 \sin 50^\circ = 6.588 \text{ km/s} \quad (a)
\]

\[
v_\perp = v \cos \gamma = 8.6 \cdot \cos 50^\circ = 5.528 \text{ km/s} \quad (b)
\]

Then equation *provides us with the angular momentum,*

\[
h = rv_\perp = 14600 \cdot 5.528 = 80710 \text{ km}^2/\text{s} \quad (c)
\]
(c) Evaluating the orbit equation at the given location on the trajectory, we get

\[ 14600 = \frac{80710^2}{398600} \frac{1}{1 + e \cos \theta} \]

From which

\[ e \cos \theta = 0.1193 \quad \text{(d)} \]

The radial component of velocity is given by equation \( v_r = \frac{\mu}{h} e \sin \theta \), so that with (a) and (c), we obtain

\[ 6.588 = \frac{398600}{80170} e \sin \theta \]

or

\[ e \sin \theta = 1.334 \quad \text{(e)} \]

Computing the ratio of (e) to (d) yields

\[ \tan \theta = \frac{1.334}{0.1193} = 11.18 \quad \Rightarrow \quad \theta = 84.89^\circ \]
(d) We substitute the true anomaly back into either (d) or (e) to find the eccentricity,

\[ e = 1.339 \]

(e) The radius of perigee can now be found from the orbit equation,

\[ r_p = \frac{h^2}{\mu} \frac{1}{1 + e \cos(0)} = \frac{80710^2}{398600} \frac{1}{1 + 1.339} = 6986 \text{ km} \]

(f) The formula for turn angle is equation \( \delta = 2 \sin^{-1}(1/e) \), from which

\[ \delta = 2 \sin^{-1}\left( \frac{1}{e} \right) = 2 \sin^{-1}\left( \frac{1}{1.339} \right) = 96.60^\circ \]

(g) The semimajor axis of the hyperbola is found in equation

\[ a = \frac{h^2}{\mu} \frac{1}{e^2 - 1} \]

(h) According to equation \( b = a\sqrt{e^2 - 1} \), the aiming radius is

\[ \Delta = a\sqrt{e^2 - 1} = 20590\sqrt{1.339^2 - 1} = 18340 \text{ km} \]